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The consequence relation in the logic of commutative *GBL*-algebras is **PSPACE**-complete

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ABSTRACT

Commutative, integral and bounded *GBL*-algebras form a subvariety of residuated lattices which provides the algebraic semantics of an interesting common fragment of intuitionistic logic and of several fuzzy logics.

It is known that both the equational theory and the quasiequational theory of commutative *GBL*-algebras are decidable (in contrast to the noncommutative case), but their complexity has not been studied yet. In this paper, we prove that both theories are in **PSPACE**, and that the quasiequational theory is **PSPACE**-hard.

1. Introduction

This paper deals with the computational complexity of a propositional logic, called GBL_{ewf}, which is a common fragment of intuitionistic logic and of several fuzzy logics. The equivalent algebraic semantics for GBL_{ewf} is given by an intensively studied variety of residuated lattices, namely commutative, integral and bounded *GBL*-algebras [13]. In this section, we introduce the system GBL_{ewf} and we discuss its logical motivations.

Basic fuzzy logic *BL* was introduced by Hájek in [11]. This logic can be regarded both as a common fragment of the three main fuzzy logics, Łukasiewicz, Gödel and product logics, as well as the logic of all continuous t-norms and their residua. A continuous t-norm $*$ is a binary continuous and weakly increasing operation on the real interval $[0, 1]$ which makes it a commutative ordered monoid with neutral element 1. The residual $\to_*$ of a continuous t-norm $*$ is uniquely determined by the condition $x * y \leq z$ if and only if $x \leq y \to_* z$. It turns out that if we interpret (multiplicative) conjunction, $\odot$, as a continuous t-norm, and implication, $\to$, as its residuum, the set of all formulas which are evaluated to 1 forms a logic, L_*, which extends *BL*. Moreover, *BL* is precisely the intersection of all logics L_* when $*$ ranges over all continuous t-norms [6]. Note that additive conjunction and disjunction are also definable in *BL* by putting $\phi \wedge \psi \;\rightleftharpoons\; \phi \odot (\phi \to \psi)$, and $\phi \vee \psi \rightleftharpoons ((\phi \to \psi) \to \psi) \wedge ((\psi \to \phi) \to \phi)$.

The intriguing observation is that neither *BL* extends intuitionistic logic *IL*, nor *IL* extends *BL*. Indeed, on the one hand, *BL* has the prelinearity axiom,

$$(\phi \to \psi) \vee (\psi \to \phi),$$

which is not provable in *IL*; and on the other hand, *IL* proves the contraction axiom,

$$\phi \to (\phi \odot \phi),$$

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which is not provable in *BL*. It is known that the minimal logic containing both *BL* and *IL* is Gödel logic (that is *IL* plus the prelinearity axiom). The question arises whether there exists an interesting common fragment of *BL* and *IL*: such an intersection would lead fuzzy interpretations of intuitionistic logic on the one hand, and constructive interpretations of fuzzy logics on the other.

A possible candidate is the logic FL_{ew}, that is, full Lambek logic plus weakening and exchange, corresponding to *IL* without contraction.[1] However, there is a principle which is common to *IL* and to *BL* and is not provable in FL_{ew}, namely the divisibility axiom:

$$(\phi \wedge \psi) \rightarrow (\phi \odot (\phi \rightarrow \psi)).$$

This principle has a nice interpretation in terms of resources: $\phi \wedge \psi$ gives you access to ϕ or to ψ up to your choice, and $\phi \rightarrow \psi$ is the weakest resource which added to ϕ gives you ψ. Thus the axiom says that your system is flexible: if you have a choice between ϕ and ψ, then you may get ϕ plus $\phi \rightarrow \psi$, so that you may always turn to ψ if you like. This observation naturally leads to the logic GBL_{ewf} (in words, *generalized basic logic plus exchange, weakening and falsum*), which is basically FL_{ew} plus the divisibility axiom, or even *BL* without prelinearity (in the latter case, $\vee$ is no longer definable in terms of $\odot$ and $\rightarrow$ and must be axiomatized as a primitive symbol).

Summarizing the discussion above, the axiomatic calculus of GBL_{ewf} is defined by the axiom schemata (A1)–(A13) and the modus ponens inference rule (R1), as follows:

(A1) $\phi \rightarrow \phi$
(A2) $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$
(A3) $(\phi \odot \psi) \rightarrow (\psi \odot \phi)$
(A4) $(\phi \odot \psi) \rightarrow \phi$
(A5) $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \odot \psi) \rightarrow \chi))$
(A6) $((\phi \odot \psi) \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \rightarrow \chi))$
(A7) $(\phi \odot (\phi \rightarrow \psi)) \rightarrow (\phi \wedge \psi)$
(A8) $(\phi \wedge \psi) \rightarrow (\phi \odot (\phi \rightarrow \psi))$
(A9) $(\phi \wedge \psi) \rightarrow (\psi \wedge \phi)$
(A10) $\phi \rightarrow (\phi \vee \psi)$
(A11) $\psi \rightarrow (\phi \vee \psi)$
(A12) $((\phi \rightarrow \psi) \wedge (\chi \rightarrow \psi)) \rightarrow ((\phi \vee \chi) \rightarrow \psi)$
(A13) $\bot \rightarrow \phi$
(R1) $\phi, \phi \rightarrow \psi \vdash_{GBL_{ewf}} \psi$

It turns out that GBL_{ewf} is strongly algebraizable in the sense of Blok and Pigozzi [4]. Its equivalent algebraic semantics is the variety of commutative, integral and bounded *GBL*-algebras (see Section 2.1 for formal definitions). As a general fact, if an algebraic variety, $\mathbb{V}$, forms the algebraic semantic of a propositional logic, *L*, in the sense of Blok and Pigozzi, then algebraic properties have a natural logical counterpart and viceversa. Indeed, the free (n-generated) algebra in the variety $\mathbb{V}$ is isomorphic to the Lindenbaum-Tarski algebra (of the n-variate fragment) of the logic *L*. In particular, the quasiequational theory of the variety $\mathbb{V}$ is equivalent to the consequence relation of the logic *L*. In the following, we adopt the algebraic view to describe the computational complexity of the consequence relation of the logic GBL_{ewf} (and related logics) in terms of the computational complexity of the quasiequational theory in the variety of commutative, integral and bounded *GBL*-algebras.

Varieties of *GBL*-algebras have been studied in [10,13,12]. In [12], it is shown that the quasiequational theory in the variety of *GBL*-algebras is undecidable, but, by contrast, quasiequations are decidable in the subvarieties of commutative *GBL*-algebras, commutative and integral *GBL*-algebras, and commutative integral and bounded *GBL*-algebras. In [2], the authors investigated the variety of hoops, corresponding to the fragment of commutative and integral *GBL*-algebras without $\bot$ and $\vee$, proving that quasiequations are decidable. However, the aforementioned papers do not contain results about the computational complexity of quasiequations in the decidable subvarieties of *GBL*-algebras. As we alluded at the beginning of this introduction, the complexity of subvarieties of commutative *GBL*-algebras, and of the corresponding propositional logics, will be the main topic of this paper.

We mentioned that the logic of commutative, integral and bounded *GBL*-algebras, GBL_{ewf}, is a common fragment of *IL* and *BL*. The computational complexity of *IL* and *BL* is known: intuitionistic validity (and consequence, via the deduction theorem) is **PSPACE**-complete [21], whereas validity and consequence in *BL* is **coNP**-complete [3], as in the classical case, despite the lack of the deduction theorem in its general form. We remark that, starting from Mundici's seminal work on Łukasiewicz logic [18], techniques based on the functional representation of free algebras have been applied for showing **coNP**-completeness of validity and consequence in fundamental schematic extensions of *BL*, namely Gödel logic and product logic. A survey of this uniform approach was given in [1].

Here, we give a *partial* complexity characterization of GBL_{ewf}. We show that the quasiequational theory of commutative, integral and bounded *GBL*-algebras (hence, the consequence problem of GBL_{ewf}) is **PSPACE**-complete (Theorem 2). In

[1] For further motivations and background on the logical counterparts of residuated lattices, we refer the reader to the recent and comprehensive monograph of Galatos et al. [9]

particular, the equational theory of commutative, integral and bounded *GBL*-algebras (hence, the validity problem of GBL_{ewf}) is in **PSPACE**, but our reduction does not generalize. We conjecture that the validity problem of GBL_{ewf} is hard for **PSPACE**.

The paper is organized as follows. In Section 2, we present the algebraic background and the combinatorial key to our problem. In Section 3, we prove our main complexity result. In Section 4, we describe some consequences of the main result.

2. Algebraic background

This section is devoted to the presentation of the algebraic background of our complexity result. In Section 2.1, we introduce some basic definitions and facts. In Section 2.2, we introduce an algebraic construction, called poset sum, that provides a complete semantics for quasiequations in commutative bounded *GBL*-algebras. In Section 2.3, we prove that, as regards to the validity of quasiequations in commutative bounded *GBL*-algebras, poset sums reduce to finite combinatorial constructions.

2.1. GBL-algebras and quasiequations

Let $(\odot, \to, \vee, \wedge, e)$ be a functional signature of type $(2, 2, 2, 2, 0)$. A *commutative residuated lattice* is a system $\mathbf{L} = (L, \odot, \to, \vee, \wedge, e)$ such that:

(i) $(L, \odot, e)$ is a commutative monoid;
(ii) $(L, \vee, \wedge)$ is a lattice;
(iii) $x \odot y \leq z$ if and only if $y \leq x \to z$ (that is, *residuation* holds).

A commutative residuated lattice is said to be *integral* if e is its top element (in this case, as is customary, we use $\top$ instead of e in the signature), *divisible* if and only if $x \leq y$ implies $y \odot (y \to x) = x$, and *bounded* if and only if it has a bottom element m and the signature has an additional constant symbol $\bot$ which is interpreted as m.

A *commutative GBL-algebra* is a divisible commutative residuated lattice. A *BL-algebra* is a commutative, integral and bounded *GBL*-algebra satisfying *prelinearity*, that is, $(x \to y) \vee (y \to x) = \top$. An *MV-algebra* is a *BL*-algebra satisfying *involutiveness* of $\neg$, that is, $\neg\neg x = x$, where $\neg x \rightleftharpoons x \to \bot$. A *Heyting algebra* is a commutative, integral and bounded *GBL*-algebra satisfying *idempotency* of $\odot$, that is, $x \odot x = x \wedge x = x$.

A *lattice ordered Abelian group* is a system $\mathbf{G} = (G, \odot, ^{-1}, \vee, \wedge, e)$ such that $(G, \odot, ^{-1}, e)$ is an Abelian group, $(G, \vee, \wedge)$ is a lattice, and $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$ (that is, $\odot$ distributes over $\vee$). Note that a lattice ordered Abelian group is a residuated lattice with respect to $\odot, \vee, \wedge, e$ by putting $x \to y \rightleftharpoons x^{-1} \odot y$. It is known that every commutative *GBL*-algebra is isomorphic to a direct product of an integral *GBL*-algebra and a lattice ordered Abelian group [10]. Therefore, since every bounded lattice ordered Abelian group is trivial, it follows that every bounded commutative *GBL*-algebra is integral.

Summarizing the previous definitions, in the sequel a system $\mathbf{A} = (A, \odot, \to, \vee, \wedge, \bot, \top)$ over the signature $\mathcal{L}_1 \rightleftharpoons (\odot, \to, \vee, \wedge, \bot, \top)$ of type $(2, 2, 2, 2, 0, 0)$ is called a commutative bounded *GBL*-algebra if: $(A, \odot, \top)$ is a commutative monoid; $(A, \vee, \wedge, \top, \bot)$ is a bounded lattice, with $\bot$ as bottom element and $\top$ as top element; $x \odot y \leq z$ if and only if $y \leq x \to z$ (that is, residuation holds); and $x \leq y$ implies $y \odot (y \to x) = x$ (that is, divisibility holds).

As already mentioned, in this paper we investigate the computational complexity of the problem of deciding if a quasiequation is valid in the variety of commutative bounded *GBL*-algebras. Let $V = \{y_j : j \in \mathbb{N}\}$ be the set of variables and $\circ \in \mathcal{L}_1 \setminus \{\top, \bot\}$. A *term* t (over $\mathcal{L}_1$) is either $\bot$, $\top$ or y_j for some $j \in \mathbb{N}$, or has the form $(t_1 \circ t_2)$, where t_1 and t_2 are terms over $\mathcal{L}_1$. Let $\mathbf{A}$ be a commutative bounded *GBL*-algebra with domain A. As is customary, a term $t(y_1, \ldots, y_l)$ with variables among $y_1, \ldots, y_l$ determines an l-ary operation $t^{\mathbf{A}}(y_1, \ldots, y_l)$ on A. With respect to pairs of terms t and s, the equation $t = s$ holds in $\mathbf{A}$ under the assignment $y_1 \mapsto a_1, \ldots, y_l \mapsto a_l$ of the variables onto elements $a_1, \ldots, a_l$ of A if and only if $t^{\mathbf{A}}(a_1, \ldots, a_l) = s^{\mathbf{A}}(a_1, \ldots, a_l)$. A *quasiequation* is an entailment statement of the form:

$$(t_1 = s_1 \text{ and } \ldots \text{ and } t_m = s_m) \text{ implies } (t = s),$$

where $m \geq 0$ and t_i, s_i, t, s are terms $(i = 1, \ldots, m)$. In a commutative residuated lattice, any statement of the form above is equivalent to the statement:

$$(u_1 \wedge e = e \text{ and } \ldots \text{ and } u_m \wedge e = e) \text{ implies } (u \wedge e = e), \tag{1}$$

where $u_i \rightleftharpoons (t_i \to s_i) \wedge (s_i \to t_i)$ for $i = 1, \ldots, m$ and $u \rightleftharpoons (t \to s) \wedge (s \to t)$. If, in addition, the commutative residuated lattice is integral, then the neutral element coincides with the top element and is denoted by $\top$, so that u_i is equivalent to $u_i \wedge \top$ $(i = 1, \ldots, m)$ and u is equivalent to $u \wedge \top$. Then, the statement above is equivalent to the statement:

$$(u_1 = \top \text{ and } \ldots \text{ and } u_m = \top) \text{ implies } (u = \top). \tag{2}$$

Both quasiequations (1) and (2) will be denoted by $(\{u_1, \ldots, u_m\}, \{u\})$ and from the context it will be clear which of (1) or (2) we are referring to. We say that a term t with variables among $y_1, \ldots, y_l$ is *valid* in a commutative bounded *GBL*-algebra $\mathbf{A}$ with domain A under the assignment $y_1 \mapsto a_1, \ldots, y_l \mapsto a_l$ of the variables onto elements $a_1, \ldots, a_l$ of A, if $t^{\mathbf{A}}(a_1, \ldots, a_l) = \top$. A quasiequation $(\{t_1, \ldots, t_m\}, \{t\})$ with variables among $y_1, \ldots, y_l$ is *valid* in $\mathbf{A}$ if and only if, for every assignment of the variables $y_1, \ldots, y_l$ onto elements of A, if $t_1, \ldots, t_m$ are valid under the assignment, then also t is. The quasiequational theory of commutative bounded *GBL*-algebras contains all and only the quasiequations valid in all the commutative bounded *GBL*-algebras.

Formally, we will study the complexity of the following decision problem, where E is a quasiequation and $\langle\cdot\rangle$ is a reasonably compact binary encoding of quasiequations:

$$\mathsf{GBL\text{-}CB\text{-}QEQ} = \{\langle E\rangle : E \text{ is valid in all commutative bounded } GBL\text{-algebras}\}.$$

We mentioned in the previous section that the logical counterpart of this algebraic question is the problem of deciding if a fixed formula ϕ is derivable in the axiomatic calculus (A1)–(A13) of GBL_{ewf} from a fixed finite set of formulae $\phi_1, \dots, \phi_m$, that is, if the finite consequence relation

$$\phi_1, \dots, \phi_m \vdash_{GBL_{ewf}} \phi$$

holds or not.

Let t be a term. Abusing notation, $|S|$ denotes the cardinality of S if S is a finite set and the length of S if S is a binary string. The number of occurrences of symbols $\odot, \to, \vee$, and $\wedge$ in t, $\mathsf{op}(t)$, is defined inductively, as follows: if $t \in \{\bot, \top\} \cup V$, then $\mathsf{op}(t) = 0$; if $t = (t_1 \circ t_2)$, then $\mathsf{op}(t) = \mathsf{op}(t_1) + \mathsf{op}(t_2) + 1$. The set of variables occurring in t, $\mathsf{var}(t)$, is defined inductively as follows: if $t \in \{\bot, \top\}$, $\mathsf{var}(t) = \emptyset$; if $t = y_j \in V$, $\mathsf{var}(t) = \{y_j\}$; if $t = (t_1 \circ t_2)$, $\mathsf{var}(t) = \mathsf{var}(t_1) \cup \mathsf{var}(t_2)$. So, $|\mathsf{var}(t)|$ is the number of distinct variables occurring in t. As is customary, for every term t, we assume a binary encoding $\langle t\rangle \in \{0, 1\}^*$ of t of length polynomial in $|\mathsf{var}(t)| + \mathsf{op}(t)$. Thus, since $|\mathsf{var}(t)| \le \mathsf{op}(t) + 1$,

$$|\langle t\rangle| \le e(\mathsf{op}(t)), \tag{3}$$

for a suitable polynomial $e : \mathbb{N} \to \mathbb{N}$. Moreover, on the basis of a reasonably compact binary encoding for sets and tuples, for any quasiequation $E = (\{t_1, \dots, t_m\}, \{t\})$, the binary encoding $\langle E\rangle \in \{0, 1\}^*$ of E has size polynomially bounded in the size of the terms $t_1, \dots, t_m, t$, that is,

$$|\langle E\rangle| \le e'(|\langle t\rangle| + \sum_{1\le i\le m} |\langle t_i\rangle|), \tag{4}$$

for a suitable polynomial $e' : \mathbb{N} \to \mathbb{N}$.

The set of *subterms* of t, $\mathsf{subt}(t)$, is defined inductively, as follows: if $t \in \{\bot, \top\} \cup V$, then $\mathsf{subt}(t) = \{t\}$; if $t = (t_1 \circ t_2)$, $\mathsf{subt}(t) = \{t\} \cup \mathsf{subt}(t_1) \cup \mathsf{subt}(t_2)$. If $T = \{t_1, \dots, t_m\}$ is a finite set of terms, then $\mathsf{var}(T) \rightleftharpoons \bigcup_{i=1}^m \mathsf{var}(t_i)$, and $\mathsf{subt}(T) \rightleftharpoons \bigcup_{i=1}^m \mathsf{subt}(t_i)$. If $E = (\{t_1, \dots, t_m\}, \{t\})$ is a quasiequation, then $\mathsf{var}(E) \rightleftharpoons \mathsf{var}(\{t_1, \dots, t_m\}) \cup \mathsf{var}(\{t\})$, and $\mathsf{subt}(E) \rightleftharpoons \mathsf{subt}(\{t_1, \dots, t_m\}) \cup \mathsf{subt}(\{t\})$.

2.2. Poset sums and finite countermodels

For any fixed integer $N \ge 1$, $[N + 1] \rightleftharpoons \{0, 1/N, \dots, (N - 1)/N, 1\}$. The basic building block of our construction is the following.

Definition 1 (*Standard MV-Chain, N-Finite MV-Chain*). Let $N \ge 1$ be a fixed integer and let $S \in \{[0, 1], [N + 1]\}$. Then, the *MV-chain* S_{MV} is the algebra of signature $\mathcal{L}_1$ defined as follows:

(i) The domain of S_{MV} is S.
(ii) The realization of $\mathcal{L}$ in S_{MV} is the following ($\circ_S$ realizes in S_{MV} the symbol $\circ$ in $\mathcal{L}$, and $x_1, x_2 \in S$):
  (ii.i) $\bot_S = 0$;
  (ii.ii) $\top_S = 1$;
  (ii.iii) $x_1 \odot_S x_2 = \max\{0, x_1 + x_2 - 1\}$;
  (ii.iv) $x_1 \vee_S x_2 = \max\{x_1, x_2\}$;
  (ii.v) $x_1 \wedge_S x_2 = \min\{x_1, x_2\}$;
  (ii.vi) $x_1 \to_S x_2 = \min\{1, -x_1 + x_2 + 1\}$.

We call $[0, 1]_{MV}$ *standard MV*-chain, and $[N + 1]_{MV}$ *N-finite MV*-chain.

Let $S \in \{[0, 1], [N+1]\}$, t be a term such that $\mathsf{var}(t) \subseteq \{y_1, \dots, y_l\}$, and $\mathbf{h} = (x_1, \dots, x_l) \in S^l$. We let $t_{\mathbf{h}}$ denote the *value* in S_{MV} of the term t under the assignment $y_j \mapsto x_j$ for $j = 1, \dots, l$, that is: if $t = y_j$, $t_{\mathbf{h}} = x_j$; if $t = \bot$, $t_{\mathbf{h}} = \bot_S$; if $t = \top$, $t_{\mathbf{h}} = \top_S$; if $t = (t_1 \circ t_2)$, $(t_1)_{\mathbf{h}}, (t_2)_{\mathbf{h}} \in S$, $t_{\mathbf{h}} = (t_1)_{\mathbf{h}} \circ_S (t_2)_{\mathbf{h}}$.

Let $b : \mathbb{N} \to \mathbb{N}$ be the polynomial defined by:

$$b(n) \rightleftharpoons 3n^3. \tag{5}$$

Lemma 1. *Let T be a finite set of terms such that* $\max_{t\in T} |\langle t\rangle| = n$, $\mathsf{var}(T) \subseteq \{y_1, \dots, y_l\}$ *and* $\mathsf{subt}(T) = \{s_1, \dots, s_m\}$, *and let* $\mathbf{a}$ *be any point in* $[0, 1]^l$. *If* $(s_1)_{\mathbf{a}} \lhd_1 (s_2)_{\mathbf{a}} \lhd_2 \cdots \lhd_{m-1} (s_m)_{\mathbf{a}}$, *where* $(\lhd_1, \dots, \lhd_{m-1}) \in \{=, <\}^{m-1}$, *then there exist* $M \le 2^{b(n)}$ *and* $\mathbf{b} \in [M + 1]^l$ *such that* $(s_1)_{\mathbf{b}} \lhd_1 (s_2)_{\mathbf{b}} \lhd_2 \cdots \lhd_{m-1} (s_m)_{\mathbf{b}}$.

Proof. The lemma is an application of [5, Proposition 3.3.1 and Proposition 9.3.3]. A *McNaughton function* over $[0, 1]^l$ is a continuous l-variate function over $[0, 1]$ such that there are l-variate linear polynomials $p_1, \dots, p_k$ with integer coefficients (the *components* of f) such that, for every $\mathbf{a} \in [0, 1]^l$, there exists $j \in \{1, \dots, k\}$ such that $f(\mathbf{a}) = p_j(\mathbf{a})$. By McNaughton's theorem [15], for every term t with $\mathsf{var}(t) \subseteq \{y_1, \dots, y_l\}$, the function $f : [0, 1]^l \to [0, 1]$ such that, for every $\mathbf{a} \in [0, 1]^l$, $f(\mathbf{a}) = t_{\mathbf{a}}$ is a McNaughton function (we say that f *corresponds* to t).

Let f_s be the l-variate McNaughton function over $[0, 1]$ corresponding to the subterm $s \in \mathsf{subt}(T)$, let $p_{s,1}, \dots, p_{s,k_s}$ be the components of f_s, and suppose that

$$\{q_1, \dots, q_k\} = \bigcup_{s \in \mathsf{subt}(T)} \{p_{s,1}, \dots, p_{s,k_s}\}.$$

For every permutation π of $\{1, \dots, k\}$, let:

$$P_\pi \rightleftharpoons \{\mathbf{a} \in [0, 1]^l : q_{\pi(1)}(\mathbf{a}) \geq q_{\pi(2)}(\mathbf{a}) \geq \cdots \geq q_{\pi(k)}(\mathbf{a})\}, \quad (6)$$
$$\mathcal{C} \rightleftharpoons \{P_\pi : P_\pi \text{ is } l\text{-dimensional}\}. \quad (7)$$

Along the lines of [5, Proposition 3.3.1], we observe that $\mathcal{C}$ is a finite set of l-dimensional polyhedra with rational vertices (that is, for every $P \in \mathcal{C}$, there exist a finite $V_P \subseteq (\mathbb{Q} \cap [0, 1])^l$ such that $P = \mathsf{conv}\, V_P$). Moreover, triangulating nonsimplicial polyhedra [8], $\mathcal{C}$ can be manufactured to a finite set $\mathcal{S}$ of l-dimensional simplexes with rational vertices (recall that an l-dimensional simplex is the convex hull of $l + 1$ vertices), having the following three properties: (i) $[0, 1]^l = \bigcup_{S \in \mathcal{S}} S$; (ii) any two simplexes in $\mathcal{S}$ intersect in a common face (as is customary, we let $\emptyset$ be the (-1)-dimensional face); (iii) for each simplex $S \in \mathcal{S}$ and $s \in \mathsf{subt}(T)$, there exists $j \in \{1, \dots, k\}$ such that the restriction of f_s to S coincides with q_j.

Now, let $\mathbf{a}$ be any point in $[0, 1]^l$, and suppose that $(s_1)_\mathbf{a} \lhd_1 (s_2)_\mathbf{a} \lhd_2 \cdots \lhd_{m-1} (s_m)_\mathbf{a}$, where $\mathsf{subt}(T) = \{s_1, s_2, \dots, s_m\}$ and $(\lhd_1, \dots, \lhd_{m-1}) \in \{=, <\}^{m-1}$. By (i)–(ii) above, there exists a face F of some simplex $S \in \mathcal{S}$ such that F is the face of S of minimal dimension containing $\mathbf{a}$. Recalling that a face of simplex is a simplex, we display the rational vertices of F as $v_1 = (c_{1,1}/d_1, \dots, c_{1,l}/d_1), \dots, v_r = (c_{r,1}/d_r, \dots, c_{r,l}/d_r)$, where $1 \leq r \leq l + 1$ and $c_{1,1}, \dots, c_{1,l}, d_1, \dots, c_{r,1}, \dots, c_{r,l}, d_r \in \mathbb{Z}$ with $0 \leq c_{1,1} \leq d_1, \dots, 0 \leq c_{1,l} \leq d_1, \dots, 0 \leq c_{r,1} \leq d_r, \dots, 0 \leq c_{r,l} \leq d_r$. Let:

$$\mathbf{b} \rightleftharpoons \left(\frac{c_{1,1} + \cdots + c_{r,1}}{d_1 + \cdots + d_r}, \dots, \frac{c_{1,l} + \cdots + c_{r,l}}{d_1 + \cdots + d_r}\right),$$

that is, let $\mathbf{b}$ be the *Farey mediant* of $v_1, \dots, v_r$. Observe that $\mathbf{b} \in (\mathbb{Q} \cap [0, 1])^l \cap F$.

We claim that $\mathbf{b}$ satisfies the statement of the lemma. Indeed, the following two facts hold. Fact 1: For every $i \neq j \in \{1, \dots, m\}$ and $\lhd \in \{<, =\}$, if $(s_i)_\mathbf{a} \lhd (s_j)_\mathbf{a}$, then $(s_i)_\mathbf{b} \lhd (s_j)_\mathbf{b}$. The case $m = 1$ is obvious. For $m > 1$, let $i = 1$ and $j = 2$ without loss of generality. Now, first suppose that $(s_1)_\mathbf{a} = (s_2)_\mathbf{a}$. By (iii), $(s_1)_\mathbf{a} = f_{s_1}(\mathbf{a}) = q_{j_1}(\mathbf{a})$ for some $j_1 \in \{1, \dots, k\}$, and $(s_2)_\mathbf{a} = f_{s_2}(\mathbf{a}) = q_{j_2}(\mathbf{a})$ for some $j_2 \in \{1, \dots, k\}$, thus $q_{j_1}(\mathbf{a}) = q_{j_2}(\mathbf{a})$. So, observing that $\mathbf{a}, \mathbf{b} \in F$ and $F \in \mathcal{S}$ is of minimal dimension such that $\mathbf{a} \in F$, by (6), $q_{j_1}(\mathbf{b}) = q_{j_2}(\mathbf{b})$. Now, by (iii), $(s_1)_\mathbf{b} = f_{s_1}(\mathbf{b}) = q_{k_1}(\mathbf{b})$ for some $k_1 \in \{1, \dots, k\}$, and $(s_2)_\mathbf{b} = f_{s_2}(\mathbf{b}) = q_{k_2}(\mathbf{b})$ for some $k_2 \in \{1, \dots, k\}$. But, since f_{s_1} and f_{s_2} are linear over F, if $f_{s_1}(\mathbf{a}) = q_{j_1}(\mathbf{a})$ and $f_{s_1}(\mathbf{b}) = q_{k_1}(\mathbf{b})$, then $q_{j_1} = q_{k_1}$ over F, and if $f_{s_2}(\mathbf{a}) = q_{j_2}(\mathbf{a})$ and $f_{s_2}(\mathbf{b}) = q_{k_2}(\mathbf{b})$, then $q_{j_2} = q_{k_2}$ over F. Summarizing, $(s_1)_\mathbf{b} = q_{k_1}(\mathbf{b}) = q_{j_1}(\mathbf{b}) = q_{j_2}(\mathbf{b}) = q_{k_2}(\mathbf{b}) = (s_2)_\mathbf{b}$, and this case is settled. The argument for proving that $(s_1)_\mathbf{a} < (s_2)_\mathbf{a}$ implies $(s_1)_\mathbf{b} < (s_2)_\mathbf{b}$ is similar. Fact 2: $\mathbf{b} \in [M+1]^l$ for some $M \leq 2^{b(n)}$. Indeed, observing that, for each subterm $s \in \mathsf{subt}(T)$, $|\langle s \rangle| \leq |\langle t \rangle|$, by [5, Proposition 9.3.3] we have that $d_1, \dots, d_r \leq 2^{4|\langle t \rangle|^2}$. Therefore,

$$d_1 + \cdots + d_r \leq r \cdot 2^{4|\langle t \rangle|^2} \leq (l + 1) \cdot 2^{4|\langle t \rangle|^2} \leq n2^{4n^2}.$$

But $n2^{4n^2} \leq 2^{b(n)}$ for every $n > 1$, thus there is $M \leq 2^{b(n)}$ such that $\mathbf{b} \in [M + 1]^l$. $\square$

A *poset* is a pair $(P, \leq_P)$ where P is a set and $\leq_P$ is binary, reflexive, antisymmetric and transitive relation over P. For any poset $(P, \leq_P)$ and any pair $(p_1, p_2) \in P^2$, we say that p_1 and p_2 are *comparable* if $p_1 \leq_P p_2$ or $p_2 \leq_P p_1$ (*incomparable* otherwise). We write $p_1 \neq_P p_2$ for distinct elements $p_1, p_2 \in P$, and $p_1 <_P p_2$, if $p_1 \leq_P p_2$ and $p_1 \neq_P p_2$. A poset $(P, \leq_P)$ is a *chain* if each pair of distinct points in P is comparable. For instance, $([0, 1], \leq)$ and $([N + 1], \leq)$ are chains, where $\leq$ denotes the order over the reals. We say that p_2 *covers* p_1 if $p_1 <_P p_2$ and there is no $q \in P$ such that $p_1 <_P q$ and $q <_P p_2$. Any poset $(P, \leq_P)$ corresponds to a directed acyclic graph (*dag*) $\mathbf{P} = (P, E_P)$, called the *cover graph* of $(P, \leq_P)$, where $E_P = \{(p_1, p_2) \in P^2 \mid p_2 \text{ covers } p_1\}$. We say that p_1 *reaches* p_2 if there exists a path from p_1 to p_2 in $\mathbf{P}$.

The following object provides the combinatorial sieve to our problem [13]. Let $\mathcal{L}_2 \rightleftharpoons (=, \leq, <)$ be a relational signature of type $(2, 2, 2)$, and let $\mathcal{L} \rightleftharpoons (\mathcal{L}_1, \mathcal{L}_2)$.

Definition 2 (*Poset Sum*). Let $\mathbf{P} = (P, E_P)$ be the cover graph of a poset $(P, \leq_P)$ and let $(\mathbf{C}_p)_{p \in P}$ be a sequence of standard *MV*-chains. The *(dual) poset sum* $\mathbf{A}$ over the *skeleton* $\mathbf{P}$ and the *summands* $(\mathbf{C}_p)_{p \in P}$, in symbols $\mathbf{A} = \bigoplus_{p \in P} \mathbf{C}_p$, is the algebra of signature $\mathcal{L}$ defined as follows (if $\circ \in \mathcal{L}$, then $\circ_p$ and $\circ_A$ are respectively for the realizations in $\mathbf{C}_p$ and $\mathbf{A}$ of the symbol $\circ$):

(i) The domain, A, of $\mathbf{A}$ is the set of all maps h on P such that:
  (i.i) for all $p \in P$, $h(p) \in \mathbf{C}_p$;
  (i.ii) for all $p \in P$, if $h(p) < \top_p$, then $\bot_q = h(q)$ for all $q \in P$ such that $q <_P p$, and (thus), if $\bot_p < h(p)$, then $h(q) = \top_q$ for all $q \in P$ such that $q >_P p$.

(ii) The realization of $\mathcal{L}$ in $\mathbf{A}$ is the following. For every $p \in P$ and $h_1, h_2 \in A$:
  (ii.i) $\bot_A(p) = \bot_p$;
  (ii.ii) $\top_A(p) = \top_p$;
  (ii.iii) $(h_1 \odot_A h_2)(p) = h_1(p) \odot_p h_2(p)$;

(ii.iv) $(h_1 \vee_A h_2)(p) = h_1(p) \vee_p h_2(p)$;
(ii.v) $(h_1 \wedge_A h_2)(p) = h_1(p) \wedge_p h_2(p)$;
(ii.vi) The realization of $\to$ in $\mathbf{A}$ is the following:
(ii.vi.i) $(h_1 \to_A h_2)(p) = h_1(p) \to_p h_2(p)$, if $h_1(q) \leq h_2(q)$ for all $q \in P$ such that $p <_P q$;
(ii.vi.ii) $(h_1 \to_A h_2)(p) = \bot_p$, otherwise;
(ii.vii) $h_1 =_A h_2$ if and only if $h_1(p) = h_2(p)$ for all $p \in P$;
(ii.viii) $h_1 \leq_A h_2$ if and only if $h_1(p) \leq h_2(p)$ for all $p \in P$;
(ii.ix) $h_1 <_A h_2$ if and only $h_1 \leq_A h_2$ and $h_1(p) < h_2(p)$ for some $p \in P$.

If P is finite, then the poset sum $\mathbf{A}$ is called *finite*. If, for all $p \in P$, $\mathbf{C}_p$ is an M-finite MV-chain for some $M \leq N$, the poset sum $\mathbf{A}$ is called N*-bounded*.

Let t be a term, $\mathbf{A}$ be a poset sum with skeleton $\mathbf{P} = (P, E_P)$ and domain A, $\mathbf{h} = (h_1, \ldots, h_l) \in A^l$, $p \in P$ and $S \in \{[0,1], [M+1]\}$ be the domain of $\mathbf{C}_p$. We let $t_{\mathbf{h},p}$ denote the value in $\mathbf{C}_p$ of the term t under the assignment $y_j \mapsto h_j(p)$ of the variables to S, $j = 1, \ldots, l$. We insist that, if $t = t_1 \to t_2$ and there exists $p <_P q$ such that $(t_2)_{\mathbf{h},q} < (t_1)_{\mathbf{h},q}$, then $t_{\mathbf{h},p} = \bot_p$ independent of the values $(t_1)_{\mathbf{h},p}, (t_2)_{\mathbf{h},p} \in S$.

Definition 3 (*Quasiequation Validity*). Let t be a term such that $\mathsf{var}(t) \subseteq \{y_1, \ldots, y_l\}$, let $\mathbf{A}$ be a poset sum with skeleton $\mathbf{P} = (P, E_P)$ and domain A, and let $\mathbf{h} = (h_1, \ldots, h_l) \in A^l$. Then: t is *valid* in $\mathbf{A}$ under $\mathbf{h}$ if, for every $p \in P$, $t_{\mathbf{h},p} = \top_p$, and we write $\mathbf{A}, \mathbf{h} \models t = \top$; otherwise, if there exists $p \in P$ such that $t_{\mathbf{h},p} < \top_p$, we say that t *fails* in $\mathbf{A}$ under $\mathbf{h}$ (with respect to p), and we write $\mathbf{A}, \mathbf{h} \not\models t = \top$.

Let $E = (\{t_1, \ldots, t_m\}, \{t\})$ be a quasiequation. A poset sum $\mathbf{A}$ *models* E, or E is *valid* in $\mathbf{A}$ (written $\mathbf{A} \models E$), if and only if the following statement holds: for every $\mathbf{h} = (h_1, \ldots, h_l) \in A^l$, if $\mathbf{A}, \mathbf{h} \models t_k = \top$ for all $k \in \{1, \ldots, m\}$, then $\mathbf{A}, \mathbf{h} \models t = \top$. If $\mathbf{A}$ does not model E, we say that $\mathbf{A}$ *falsifies* E, or that $\mathbf{A}$ is a *countermodel* to E, or that E *fails* in $\mathbf{A}$ (written $\mathbf{A} \not\models E$). In this case, if $\mathbf{h} \in A^l$ and $p \in P$ are such that $\mathbf{A}, \mathbf{h} \models t_k = \top$ for all $k \in \{1, \ldots, m\}$, but $t_{\mathbf{h},p} < 1$, we say that E *fails* in $\mathbf{A}$ with respect to $\mathbf{h}$ and p.

Example 1. Let $\mathbf{P} = (P, E_P)$ be the poset over $P = \{1, 2, 3, 4\}$ given by $1 <_P 2 <_P 4 >_P 3 >_P 1$, and let $\mathbf{A}$ be the finite poset sum having $\mathbf{P}$ as skeleton and a sequence of standard MV-chains indexed by P as summands. It is possible to check that $\mathbf{A}$ is a commutative bounded GBL-algebra.

Let y_1 and y_2 be distinct variables, and let $\mathbf{h} = (h_1, h_2) \in A^2$ be such that: $h_1(1) = h_2(1) = 0$; $h_1(2) = 1/2$ and $h_2(2) = 0$; $h_1(3) = 0$ and $h_2(3) = 1$; $h_1(4) = h_2(4) = 1$. For all $p \in P$, $(y_1)_{\mathbf{h},p} = h_1(p)$ and $(y_2)_{\mathbf{h},p} = h_2(p)$. By Definition 2:

p	$(y_1)_{\mathbf{h},p}$	$(y_2)_{\mathbf{h},p}$	$(y_1 \odot y_1)_{\mathbf{h},p}$	$(y_1 \to y_2)_{\mathbf{h},p}$	$(y_2 \to y_1)_{\mathbf{h},p}$	$((y_1 \to y_2) \vee (y_2 \to y_1))_{\mathbf{h},p}$
1	0	0	0	0	0	0
2	1/2	0	0	1/2	1	1
3	0	1	0	1	0	1
4	1	1	1	1	1	1

Idempotency fails in $\mathbf{A}$ with respect to $\mathbf{h}$ and 2, because $(y_1 \odot y_1)_{\mathbf{h},2} = 0 < 1/2 = (y_1)_{\mathbf{h},2}$; also, prelinearity fails in $\mathbf{A}$ with respect to $\mathbf{h}$ and 1, because $((y_1 \to y_2) \vee (y_2 \to y_1))_{\mathbf{h},1} = 0 < 1 = \top_1$. Thus, $\mathbf{A}$ is neither a BL-algebra nor a Heyting algebra.

Our main result relies on a sharpening of the following characterization [13].

Theorem 1 (*Jipsen and Montagna*). *Let E be a quasiequation. Then, $\langle E \rangle \notin$ GBL-CB-QEQ if and only if there exists a finite poset sum $\mathbf{A}$ such that $\mathbf{A} \not\models E$.*

In the next section, we will sharpen the previous statement, proving that if E fails in a commutative bounded GBL-algebra, then E already fails in a finite poset sum with skeleton and summands explicitly bounded in the size of E.

2.3. Countermodel bounds

In this section, we prove that if a quasiequation E of size n fails in a finite poset sum, then E fails in a finite poset sum having a tree of height polynomial in n and cardinality exponential in n as skeleton, and chains of cardinality exponential in n as summands. Observing that the converse clearly holds, this sharpens the statement of Theorem 1.

For sake of conciseness, we first fix some specialized terminology and notation relative to finite poset sums. Let E be a quasiequation, t be a term in $\mathsf{subt}(E)$, $\mathbf{A}$ be a finite poset sum specified as in Definition 2, $\mathbf{h} = (h_1, \ldots, h_l) \in A^l$ and $p \in P$.

If $\bot_p < t_{\mathbf{h},p}$, then for every $q \in P$ such that $p <_P q$, the value $t_{\mathbf{h},q}$ is *forced* to be equal to $\top_q$, so that the value of t on p (under $\mathbf{h}$) acts as a constraint on the value of t (under $\mathbf{h}$) on *every* $q \in P$ such that $p <_P q$. For this reason, we say that t is *hibernated* on p (under $\mathbf{h}$), and that t *propagates* a *universal* constraint above p, or in short *is* a universal constraint on p (under $\mathbf{h}$). We write $\mathsf{subt}(E)_{(\mathbf{h},p,\top)} \subseteq \mathsf{subt}(E)$ for the set of subterms of E hibernated on p (under $\mathbf{h}$).

Let $\mathsf{subt}(E)_{\to} \subseteq \mathsf{subt}(E)$ be the set of implicative subterms of E. If $t = t_1 \to t_2 \in \mathsf{subt}(E)_{\to}$, and every node q such that $p <_P q$ satisfies the constraint $(t_1)_{\mathbf{h},q} \leq (t_2)_{\mathbf{h},q}$, in light of Definition 2(ii.vi.i), we say that t is evaluated *pointwise* on p (under $\mathbf{h}$); otherwise, if there is a node q such that $p <_P q$ satisfying the constraint $(t_1)_{\mathbf{h},q} > (t_2)_{\mathbf{h},q}$, in light of Definition 2(ii.vi.ii), we say that t is *not* evaluated pointwise on p (under $\mathbf{h}$). In other words, if we know that t is evaluated pointwise on p (under $\mathbf{h}$), then we can take for granted that *every* node q above p in the poset satisfies the constraint $(t_1)_{\mathbf{h},q} \leq (t_2)_{\mathbf{h},q}$. For this reason, if t is evaluated pointwise on p (under $\mathbf{h}$), we say that t *propagates* a *universal* constraint above p (under $\mathbf{h}$), or in short, that t *is* a universal constraint on p (under $\mathbf{h}$). We write $\mathsf{subt}(E)_{(\to,\mathbf{h},p,\forall)} \subseteq \mathsf{subt}(E)_{\to}$ for the set of implicative subterms of E evaluated pointwise on p. Conversely, if we know that t is *not* evaluated pointwise on p (under $\mathbf{h}$), then we can take for granted that there *exists* a node $p <_P q \in P$ that satisfies the constraint $(t_1)_{\mathbf{h},q} > (t_2)_{\mathbf{h},q}$. For this reason, we say that t *generates* an *existential* constraint above p (under $\mathbf{h}$), or in short, that t *is* an existential constraint on p (under $\mathbf{h}$). We write $\mathsf{subt}(E)_{(\to,\mathbf{h},p,\exists)} \subseteq \mathsf{subt}(E)_{\to}$ for the set of implicative subterms of E not evaluated pointwise on p.

Let v be any existential constraint on p (under $\mathbf{h}$), that is, $v = v_1 \to v_2 \in \mathsf{subt}(E)_{(\to,\mathbf{h},p,\exists)}$. Let r be any node in P reachable from p. If there is no node $q \in P$ such that $p <_P q <_P r$ satisfying $(v_1)_{\mathbf{h},q} > (v_2)_{\mathbf{h},q}$, then we say that r *inherits* the existential constraint on v. If r is a *maximal* element in $\mathbf{P}$ such that $p <_P r$ and $(v_1)_{\mathbf{h},r} > (v_2)_{\mathbf{h},r}$, then we say that r *fixes* the existential constraint on v (generated by p). Let u be any universal constraint on p (under $\mathbf{h}$), that is, $u \in \mathsf{subt}(E)_{(\to,\mathbf{h},p,\forall)} \cup \mathsf{subt}(E)_{(\mathbf{h},p,\top)}$. If $r \in P$ and $p <_P r$, we say that r *inherits* the universal constraint on u.

Example 2. Let $\mathbf{A}$ and $\mathbf{h}$ be settled as in Example 1, and let E be a quasiequation with $y_1, y_2, \top, y_1 \to y_2, y_2 \to y_1, (y_1 \to y_2) \vee (y_2 \to y_1) \in \mathsf{subt}(E)$.

Terms $y_1, \top, y_1 \to y_2, y_2 \to y_1, (y_1 \to y_2) \vee (y_2 \to y_1)$ are in $\mathsf{subt}(E)_{(\mathbf{h},2,\top)}$, because their values on node 2 are strictly greater than the value of $\perp_2$; their values on node $4 >_P 2$ are equal to the value of $\top_4$. Similarly, $\{y_2, \top, y_1 \to y_2, (y_1 \to y_2) \vee (y_2 \to y_1)\} \subseteq \mathsf{subt}(E)_{(\mathbf{h},3,\top)}$.

Term $y_1 \to y_2$ is in $\mathsf{subt}(E)_{(\to,\mathbf{h},1,\exists)}$, in fact $2 >_P 1$ and $(y_1)_{\mathbf{h},2} > (y_2)_{\mathbf{h},2}$; thus, node 1 generates an existential constraint on $y_1 \to y_2$, which is inherited and fixed by node 2 (node 3 inherits but does not fixes it). Term $y_2 \to y_1$ is in $\mathsf{subt}(E)_{(\to,\mathbf{h},1,\exists)}$, in fact $3 >_P 1$ and $(y_2)_{\mathbf{h},3} > (y_1)_{\mathbf{h},3}$; thus, node 1 generates an existential constraint on $y_2 \to y_1$, which is inherited and fixed by node 3 (node 2 inherits but does not fixes it).

Term $y_1 \to y_2$ is in $\mathsf{subt}(E)_{(\to,\mathbf{h},2,\forall)}$, because $(y_1)_{\mathbf{h},4} \leq (y_2)_{\mathbf{h},4}$; thus, node 2 propagates a universal constraint on $y_1 \to y_2$, which is inherited by node 4. Term $y_2 \to y_1$ is in $\mathsf{subt}(E)_{(\to,\mathbf{h},3,\forall)}$, because $(y_1)_{\mathbf{h},4} \leq (y_2)_{\mathbf{h},4}$; thus, node 3 propagates a universal constraint on $y_2 \to y_1$, which is inherited by node 4.

Adopting the above terminology and notation, we provide explicit bounds on the size of finite countermodels to quasiequations. Let $q : \mathbb{N} \to \mathbb{N}$ be the polynomial defined by:

$$q(n) \risingdotseq n^2. \tag{8}$$

Lemma 2. *Let $E = (\{t_1, \ldots, t_m\}, \{t\})$ be a quasiequation of size n, and let $\mathbf{A}$ be a finite poset sum with skeleton $\mathbf{P} = (P, E_P)$ where E fails. Then, there exists a finite $2^{b(n)}$-bounded poset sum $\mathbf{B}$ where E fails, such that the skeleton of $\mathbf{B}$ is a rooted tree $\mathbf{T} = (T, E_T)$, of height at most n and cardinality at most $2^{q(n)}$.*

Proof. Let $\mathsf{var}(E) = \{y_1, \ldots, y_l\}$, let A be the domain of $\mathbf{A}$, and let $\mathbf{h} = (h_1, \ldots, h_l) \in A^l$ and $p \in P$ be such that E fails in $\mathbf{A}$ with respect to $\mathbf{h}$ and p. We prove that there exists a poset sum $\mathbf{B}$ satisfying the statement of the lemma.

The skeleton $\mathbf{T} = (T, E_T)$ of $\mathbf{B}$ is a rooted tree, defined as follows. The root of $\mathbf{T}$ is a node $v(p)$ corresponding to the node $p \in P$. Recall that, if $\mathsf{subt}(E)_{(\to,\mathbf{h},p,\exists)}$ is not empty, then the node $p \in P$ generates (in $\mathbf{A}$) existential constraints on each term in $\mathsf{subt}(E)_{(\to,\mathbf{h},p,\exists)}$. Let $v(q)$ be a node in T, corresponding to the node $q \in P$. There are two cases. Case 1: $\mathsf{subt}(E)_{(\to,\mathbf{h},q,\exists)} = \emptyset$. In this case, $v(q)$ is a leaf of $\mathbf{T}$. Case 2: $\mathsf{subt}(E)_{(\to,\mathbf{h},q,\exists)} \neq \emptyset$. In this case, the only edges leaving $v(q)$ in $\mathbf{T}$ are $(v(q), v(r_1)), \ldots, (v(q), v(r_k)) \in E_T$, where $v(r_1), \ldots, v(r_k) \in T$ are nodes of $\mathbf{T}$, corresponding to nodes $r_1, \ldots, r_k \in P$ respectively, satisfying the following:

(T1) for $i = 1, \ldots, k$, r_i is *reachable* from q in $\mathbf{P}$;
(T2) for $i = 1, \ldots, k$, there exists $s \in \mathsf{subt}(E)_{(\to,\mathbf{h},q,\exists)}$ such that r_i is the *only* node in $\{r_1, \ldots, r_k\}$ that *fixes* s;
(T3) the union of the terms fixed by r_1, the terms fixed by $r_2, \ldots$, and the terms fixed by r_k, is exactly $\mathsf{subt}(E)_{(\to,\mathbf{h},q,\exists)}$.

We remark that $r_1, \ldots, r_k$ are pairwise distinct by (T2), but there may be distinct nodes in T corresponding to the same node in P. The intuition underlying conditions (T1)–(T3) is that the covers of $v(q)$ are exactly those nodes that are necessary and sufficient, by (T2) and (T3) respectively, to fix all the existential constraints pending on $v(q)$. Notice that nodes $r_1, \ldots, r_k$ satisfying (T1)–(T3) exist in $\mathbf{P}$. Indeed, $\mathbf{A}$ respects Definition 2, so that there exists a collection W of nodes $w_1, \ldots, w_o >_P q$ satisfying (T3); on the basis of W, compute a collection W' satisfying (T2) inductively, as follows: $W_0 \risingdotseq W$; for $1 \leq j \leq o$: $W_j \risingdotseq W_{j-1} \setminus \{w_j\}$ if all the terms fixed by w_j are already fixed by a node in $W_{j-1} \setminus \{w_j\}$, otherwise $W_j \risingdotseq W_{j-1}$; $W' \risingdotseq W_o$.

Claim 1. $\mathbf{T}$ *has height at most n and cardinality at most $2^{q(n)}$.*

Proof. First, we observe that every leaf of $\mathbf{T}$ has depth at most n. Indeed, let $q \in P$ be such that no edge leaving $v(q)$ is in E_T (that is, $v(q)$ is a leaf of $\mathbf{T}$), and suppose, for contradiction, that $v(q)$ has depth greater than n in $\mathbf{T}$. W.l.o.g., let the depth of $v(q)$ be equal to $n + 1$. Then, there exists in $\mathbf{T}$ a path $(v(p) = v(r_0), v(r_1), \ldots, v(r_n), v(r_{n+1}) = v(q))$ from $v(p)$ to $v(q)$ of length $n + 1$. Each edge (r_i, r_{i+1}), $0 \leq i < n$, corresponds to the fact that r_{i+1} fixes some $s \in \mathsf{subt}(E)_{(\to,r_i,\mathbf{h},\exists)}$, and, since there are at most $|\mathsf{subt}(E)| \leq n$ distinct subterms, there must be a subterm s fixed twice, once by r_i and next by r_j, for some $0 \leq i < j \leq n + 1$. This, by definition, observed that $r_i <_P r_j$, contradicts the assumption that r_i fixes s. Thus, any leaf of $\mathbf{T}$ has depth $\leq n$, so $\mathbf{T}$ has height at most n.

Second, we observe that every internal node of $\mathbf{T}$ has degree at most n. Indeed, let $q \in P$ and suppose that the edges leaving $v(q)$ in E_T are exactly $(v(q), v(r_1)), \ldots, (v(q), v(r_k))$. By construction, $\mathsf{subt}(E)_{(\to,q,\mathbf{h},\exists)} \neq \emptyset$ and, for all $i = 1, \ldots, k$, there exists $s \in \mathsf{subt}(E)_{(\to,q,\mathbf{q},\exists)}$ such that r_i is the only node in $\{r_1, \ldots, r_k\}$ that fixes s. But, since there are at most $|\mathsf{subt}(E)| \leq n$ subterms in $\mathsf{subt}(E)_{(\to,q,\mathbf{h},\exists)}$, there are at most n edges in $\mathbf{T}$ leaving $v(q)$ (that is, $k \leq n$). Therefore, the cardinality $|T|$ of $\mathbf{T}$ is bounded above by the number of nodes of a complete n-ary tree of height n (a rooted tree in which all leaves have depth n and all internal nodes have degree n), that is, $|T| \leq n^{n+1} \leq n2^{n \log_2 n}$. Since, $n2^{n \log_2 n} \leq 2^{q(n)}$ for every $n \geq 1$, the cardinality of $\mathbf{T}$ at most $2^{q(n)}$.

This settles the claim. $\square$

The previous claim addressed the skeleton of $\mathbf{B}$. Now we handle the summands of $\mathbf{B}$.

Claim 2. *For every $v(q) \in T$, there exists $M_{v(q)} \leq 2^{b(n)}$ such that, letting*

$$\mathbf{B} \rightleftharpoons \bigoplus_{v(q)\in T} [M_{v(q)} + 1]_{MV},$$

E fails in $\mathbf{B}$.

Proof. Let $\mathbf{A}'$ be the poset sum having $\mathbf{T}$ as skeleton and standard MV-chains $[0, 1]_{MV}$ as summands.

First observe that E fails in $\mathbf{A}'$ with respect to the (root) node $v(p) \in T$, corresponding to $p \in P$, and the assignment $\mathbf{h}' = (h'_1, \ldots, h'_l) \in (A')^l$ such that $h'_1(v(q)) = h_1(q), \ldots, h'_l(v(q)) = h_l(q)$, where $v(q)$ is a node in T and q is the node in P such that $v(q)$ corresponds to q. This observation holds since, by (T1)–(T3), $u_{\mathbf{h},q} = u_{\mathbf{h}',v(q)}$ for every $q \in P$ and every $u \in \mathsf{subt}(E)$, where $v(q)$ is any node in T corresponding to the node q in P. Then, we have that $\mathbf{A}', \mathbf{h}' \models t_i = \top$ for $i = 1, \ldots, m$ but, since E fails in $\mathbf{A}$ with respect to $\mathbf{h}$ and p,

$$t_{\mathbf{h}',v(p)} < \top_{v(p)} = (t_1)_{\mathbf{h}',v(p)} = \cdots = (t_m)_{\mathbf{h}',v(p)}.$$

Let $v(q)$ be a node of $\mathbf{T}$, q be the node of $\mathbf{P}$ such that $v(q)$ corresponds to q, let $\mathsf{subt}(E) = \{s_1, \ldots, s_r\}$ be the subterms of E (by definition $t, t_1, \ldots, t_m \in \mathsf{subt}(E)$), and let $(\lhd_1, \ldots, \lhd_r) \in \{<, =\}^r$ be such that the chain,

$$(s_1)_{\mathbf{h}',v(q)} \lhd_1 \cdots \lhd_{r-1} (s_r)_{\mathbf{h}',v(q)} \lhd_r \top_{v(q)}, \tag{9}$$

holds in $\mathbf{A}'$. The idea is the following. On the basis of $\mathbf{h}' \in (A')^l$, we compute an integer $M_{v(q)} \leq 2^{b(n)}$ and an assignment $(k_1(v(q)), \ldots, k_l(v(q))) \in [M_{v(q)} + 1]^l$ that respects (9). Eventually we obtain $\mathbf{k} = (k_1, \ldots, k_l) \in B^l$ such that E fails in $\mathbf{B}$ with respect to $\mathbf{k}$. We examine two cases.

Case 1: Suppose that all the subterms of E of the form $u_1 \to u_2$ are evaluated pointwise in $\mathbf{A}'$ with respect to $\mathbf{h}'$ and $v(q)$. Then, letting $\mathbf{a} \rightleftharpoons (h'_1(v(q)), \ldots, h'_l(v(q))) \in [0, 1]^l$, we have

$$(s_1)_{\mathbf{a}} \lhd_1 \cdots \lhd_{r-1} (s_r)_{\mathbf{a}} \lhd_r \top_{[0,1]}.$$

Noting that, by (3) and (4), $\max_{u\in\mathsf{subt}(E)} |\langle u\rangle| \leq n$, by Lemma 1, there exist $M_{v(q)} \leq 2^{b(n)}$ and $\mathbf{b} = (b_1, \ldots, b_l) \in [M_{v(q)} + 1]^l$ such that

$$(s_1)_{\mathbf{b}} \lhd_1 \cdots \lhd_{r-1} (s_r)_{\mathbf{b}} \lhd_r \top_{[M_{v(q)}+1]}.$$

Letting $k_1(v(q)) \rightleftharpoons b_1, \ldots, k_l(v(q)) \rightleftharpoons b_l$, we have that

$$(s_1)_{\mathbf{k},v(q)} \lhd_1 \cdots \lhd_{r-1} (s_r)_{\mathbf{k},v(q)} \lhd_r \top_{v(q)},$$

holds in the poset sum $\mathbf{B}$ having as its $(v(q))$th summand the MV-chain $[M_{v(q)} + 1]_{MV}$. In particular, $u_{\mathbf{k},v(q)} = \top_{v(q)}$ for every $u \in \{t_1, \ldots, t_m\}$, and $t_{\mathbf{k},v(q)} < \top_{v(q)}$ if $v(q) = v(p)$. This settles the first case.

Case 2: Now suppose the contrary, and let $W = \{w_1, \ldots, w_k\}$ be the subterms of E of the form $u_1 \to u_2$ not evaluated pointwise in $\mathbf{A}'$ with respect to $\mathbf{h}'$ and $v(q)$, and suppose that $\mathsf{op}(w_1) \geq \cdots \geq \mathsf{op}(w_k)$. By Definition 2(ii.vi.ii), $w_{\mathbf{h}',v(q)} = \bot_{v(q)}$ for every $w \in W$. For every $s \in \mathsf{subt}(E)$, let s' be the term obtained by substituting sequentially first w_1 with $\bot$ in s, then w_2 with $\bot$ in $s[w_1 \leftarrow \bot], \ldots,$ finally w_k with $\bot$ in $s[w_1 \leftarrow \bot, \ldots, w_{k-1} \leftarrow \bot]$. Observe that $s_{\mathbf{h}',v(q)} = (s')_{\mathbf{h}',v(q)}$ in $\mathbf{A}'$, therefore we have that

$$(s'_1)_{\mathbf{h}',v(q)} \lhd_1 \cdots \lhd_{r-1} (s'_r)_{\mathbf{h}',v(q)} \lhd_r \top_{v(q)},$$

holds in $\mathbf{A}'$. Then, letting $\mathbf{a} \rightleftharpoons (h'_1(v(q)), \ldots, h'_l(v(q))) \in [0, 1]^l$, we have

$$(s'_1)_{\mathbf{a}} \lhd_1 \cdots \lhd_{r-1} (s'_r)_{\mathbf{a}} \lhd_r \top_{[0,1]}.$$

Noting that, by (3) and (4), $\max_{u\in\mathsf{subt}(E)} |\langle u'\rangle| \leq n$, by Lemma 1 there exist $M_{v(q)} \leq 2^{b(n)}$ and $\mathbf{b} = (b_1, \ldots, b_l) \in [M_{v(q)} + 1]^l$ such that

$$(s'_1)_{\mathbf{b}} \lhd_1 \cdots \lhd_{r-1} (s'_r)_{\mathbf{b}} \lhd_r \top_{[M_{v(q)}+1]}.$$

Letting $k_1(v(q)) \rightleftharpoons b_1, \ldots, k_l(v(q)) \rightleftharpoons b_l$, we have that

$$(s'_1)_{\mathbf{k},v(q)} \lhd_1 \cdots \lhd_{r-1} (s'_r)_{\mathbf{k},v(q)} \lhd_r \top_{v(q)},$$

holds in the poset sum $\mathbf{B}$ having as its $(v(q))$th summand the MV-chain $[M_{v(q)} + 1]_{MV}$. In particular, $u'_{\mathbf{k},v(q)} = \top_{v(q)}$ for every $u \in \{t_1, \ldots, t_m\}$, and $t'_{\mathbf{k},v(q)} < \top_{v(q)}$ if $v(q) = v(p)$. But, for every $s \in \mathsf{subt}(E)$, $s'_{\mathbf{k},v(q)} = s_{\mathbf{k},v(q)}$ in $\mathbf{B}$. This settles the second case.

By the previous two cases, we have that for every $v(q) \in T$ there exists an integer $M_{v(q)} \leq 2^{b(n)}$ and an assignment $(k_1(v(q)), \dots, k_l(v(q))) \in [M_{v(q)} + 1]^l$ that respects (9). Thus, since we observed in the beginning that E fails in $\mathbf{A}'$ with respect to the root $v(p) \in T$ and the assignment $\mathbf{h}' = (h'_1, \dots, h'_l) \in (A')^l$, we conclude that E fails in $\mathbf{B}$ with respect to the root $v(p) \in T$ and the assignment $\mathbf{k} = (k_1, \dots, k_l) \in B^l$ described above.

Since $\mathbf{B}$ is $2^{b(n)}$-bounded by construction, this settles the claim. $\square$

By the previous two claims, $\mathbf{B}$ is in fact the required poset sum, and the lemma is proved. $\square$

In the next section, we will prove that, given a quasiequation E of size n, if E fails in some commutative bounded *GBL*-algebra, then it is possible to guess a countermodel $\mathbf{B}$ to E determined as in the statement of Lemma 2, using a polynomial amount of memory space.

3. Quasiequations complexity

This section is devoted to the presentation of our main complexity result.

The algorithm we present below decides the complement of the problem GBL-CB-QEQ, written $\overline{\text{GBL-CB-QEQ}}$, that is, on input a quasiequation E, the output is 1 if and only if E is not valid. Intuitively, the algorithm guesses a countermodel to E, such that there is a succeeding guess if and only if E is not valid. The model of computation we adopt is the following.

Definition 4. An *online (nondeterministic) Turing machine*, is a deterministic Turing machine having a two-way read-only input tape, a two-way read-write work tape, and a unidirectional read-only guess tape. The content of the guess tape is selected nondeterministically. The machine *accepts* the input string x if there exists a guess string y such that, when the machine starts working with x on the input tape and y on the guess tape, it eventually enters an accepting state.

So, in this model of computation, only the space used on the work tape is metered. It is known that, with respect to decision problems, online Turing machines are (time and) space equivalent to standard nondeterministic Turing machines with a two-way read-only input tape and a two-way read-write work tape.

Definition 5. A decision problem X is in **NPSPACE** if there exists an online Turing machine M such that, for any binary input string x:

(i) there exists a binary guess string y such that M accepts working on (x, y) if and only if $x \in X$;
(ii) for any guess string y, $M(x, y)$ uses an amount of space bounded above by a polynomial in $|x|$.

The present section is organized as follows. In Section 3.1, we describe the algorithm, called GuessCountermodel, and we prove that the algorithm decides the problem $\overline{\text{GBL-CB-QEQ}}$ (Lemma 3) and works in polynomial space (Lemma 4). Thus, GBL-CB-QEQ $\in$ **PSPACE**. In Section 3.2 we prove that GBL-CB-QEQ is hard for **PSPACE** (Lemma 5). Our main result follows:

Theorem 2. GBL-CB-QEQ *is* **PSPACE**-*complete.*

For background on algorithms and complexity we refer to [7] and [19].

3.1. Upper bound

In this section, we describe a polynomial-space decision algorithm for the problem $\overline{\text{GBL-CB-QEQ}}$: on input a quasiequation E of size n, the algorithm outputs 1 if and only if E is not valid, using an amount of memory space polynomial in n.

Recall that, by Lemma 2, if the quasiequation $E = (\{t_1, \dots, t_m\}, \{t\})$ of size n over variables $\{y_1, \dots, y_l\}$ is not valid, there exists a finite $2^{b(n)}$-bounded poset sum $\mathbf{B}$ having as skeleton a (rooted) tree $\mathbf{T} = (T, E_T)$, of height at most n and cardinality at most $2^{q(n)}$, such that E fails in $\mathbf{B}$ with respect to some $\mathbf{k} = (k_1, \dots, k_l) \in B^l$ and the root r of T. In the nondeterministic framework of Definition 4, it is possible to guess $\mathbf{B}$ and $\mathbf{k}$, and check that E fails in $\mathbf{B}$ with respect to $\mathbf{k}$ and r. But, since we aim to a polynomial space algorithm, in light of Definition 5(ii) it is not possible to store in memory the whole of the structure $\mathbf{B}$ or the whole of the assignment $\mathbf{k}$, because these objects have size exponential in n. Nevertheless, we will show that it is possible to guess $\mathbf{B}$ and $\mathbf{k}$ iteratively, using an amount of memory space polynomial in n. The idea is the following (some details, here omitted in the interest of readability, will be made explicit by the pseudocode).

Initialization (Step $b = 1$). The algorithm creates a node x, and then guesses the following information: first, a positive integer $M_x \leq 2^{b(n)}$ (intuitively, the cardinality of the *MV*-chain corresponding to x); second, a tuple $\mathbf{x} = (x_1, \dots, x_l) \in [M_x + 1]^l$ (intuitively, the assignment $y_1 \mapsto x_1, \dots, y_l \mapsto x_l$ of variables in $\mathsf{var}(E)$ over the *MV*-chain corresponding to x); third, a pair $S_x = (S_{x,\forall}, S_{x,\exists})$ where $S_{x,\forall}, S_{x,\exists} \subseteq \mathsf{subt}(E)$ (intuitively, $S_{x,\forall}$ is $\mathsf{subt}(E)_{(\rightarrow,\mathbf{h},p,\forall)} \cup \mathsf{subt}(E)_{(\mathbf{h},p,\top)}$ and $S_{x,\exists}$ is $\mathsf{subt}(E)_{(\rightarrow,\mathbf{h},p,\exists)}$, so that S_x contains universal and existential constraints on node x with respect to $\mathbf{x}$). At this stage, the algorithm checks if the assignment $\mathbf{x}$ is *sound*, that is, if $\mathbf{x}$ extends to a valuation of the subterms in $\mathsf{subt}(E)$ such that $t_{\mathbf{x}} < \top_{[M_u+1]} = 1 = (t_1)_{\mathbf{x}} = \dots = (t_m)_{\mathbf{x}}$ holds. If $\mathbf{x}$ is not sound, the algorithm outputs 0. Otherwise, the algorithm stores S_x in memory, so that the allocation amounts to the list (S_x). Intuitively, the algorithm memorizes that every node reachable from x must satisfy all the universal constraints on x, and possibly may satisfy some existential constraint on x. The node x is distinguished as the only node with no parent, so we call it *root*. At step $b + 1$, (x) will be referenced as the *current* path, x as the *current* node, and S_x as the *pendings* on x.

Iteration (Step $2 \le b \le 2^{q(n)+1} - 1$*).* If $b = 2^{q(n)+1} - 1$, the algorithm outputs 0. Otherwise, let $(x, \dots, w, v)$ be the current path, v be the current node, with parent w (the case where v is the root is treated as an exception), and let S_v be the pendings on v. There are two cases. Case 1: $S_{v,\exists} \ne \emptyset$. In this case, the algorithm creates a new node u, having v as parent, and guesses the following information (as above): $M_u \le 2^{b(n)}$, $\mathbf{u} \in [M_u + 1]^l$, and $S_u = (S_{u,\forall}, S_{u,\exists})$. At step $b + 1$, $(u, \dots, w, v, u)$, u, and S_u respectively, will be referenced as the current path, node and pendings. At this stage, the algorithm checks if the assignment $\mathbf{u}$ is *sound*, that is, if $\mathbf{u}$ satisfies all the inherited universal constraints and, in addition, at least one inherited existential constraint. If $\mathbf{u}$ is not sound, the algorithm outputs 0. At the implementation level, the soundness of $\mathbf{u}$ reduces to satisfiability of a certain finite set of linear equality and inequality constraints, as specified in detail in the pseudocode. For instance, if $(t_1)_{\mathbf{u}} = \cdots = (t_m)_{\mathbf{u}} = \top_{[M_u+1]} = 1$ does not hold, the algorithm outputs 0. If $\mathbf{u}$ is sound, the algorithm updates $S_x, \dots, S_w$ by removing every term $s = s_1 \to s_2 \in S_v$ corresponding to an existential constraint that is satisfied by u under $\mathbf{u}$ (that is, such that $(s_2)_{\mathbf{u}} < (s_1)_{\mathbf{u}}$ holds); then the algorithm stores S_u, so that the allocation amounts to $(S_x, \dots, S_w, S_v, S_u)$, and eventually executes the $(b+1)$th step. Case 2: $S_{v,\exists} = \emptyset$. If v is the root, the algorithm outputs 1. Otherwise, the algorithm backtracks to w (at step $b+1$, $(x, \dots, w)$, w, S_w respectively will be referenced as the current path, node and pendings) and executes the $(b + 1)$th step.

The intuition underlying the process is the following. If E is not valid, we know that E fails in a poset sum $\mathbf{B}$ specified as in Lemma 2. Let $\mathbf{k}$ be the assignment such that E fails in $\mathbf{B}$ under $\mathbf{k}$. The described algorithm is intended to simulate a preorder traversal of $\mathbf{T}$, starting the visit from the root r (and storing only the path from the last visited node to r). For every visited node v, the algorithm is intended to guess the assignment $\mathbf{k} \restriction v \risingdotseq (k_1(v), \dots, k_l(v)) \in [2^{b(n)} + 1]^l$. Clearly, the assignment $\mathbf{k} \restriction r$ satisfies the constraint $t_{\mathbf{k}\restriction r} < 1 = (t_1)_{\mathbf{k}\restriction r} = \cdots = (t_m)_{\mathbf{k}\restriction r}$ with respect to the root node r, and the assignment $\mathbf{k} \restriction v$ satisfies the constraint $1 = (t_1)_{\mathbf{k}\restriction v} = \cdots = (t_m)_{\mathbf{k}\restriction v}$ with respect to every node $v \ne_T r$. Moreover, for every node $v \in T$, the assignment $\mathbf{k} \restriction v$ satisfies the universal constraints inherited by v and also, if v generates an existential constraint for a term $s_1 \to s_2$, then there is a node $w \in T$ covering v (recall (T2) above) such that $\mathbf{k} \restriction w$ satisfies $(s_1)_{\mathbf{k}\restriction w} > (s_2)_{\mathbf{k}\restriction r}$. The traversal of $\mathbf{T}$ terminates in at most $2^{q(n)+1} - 1$ steps (in fact, such a number of steps suffices to traverse a complete n-ary tree of height n), the last visited node is the root r of $\mathbf{T}$, and condition $S_{r,\exists} = \emptyset$ holds. Thus, the algorithm outputs 1. Conversely, if the algorithm outputs 1, then there is a successful sequence of guesses that, modulo details to be specified, corresponds to a poset sum $\mathbf{B}$ and an assignment $\mathbf{k}$ as above such that E fails in $\mathbf{B}$ with respect to $\mathbf{k}$ and the root r of $\mathbf{T}$.

The pseudocode listed below, modularized into a main procedure, GuessCountermodel, and two subprocedures, GuessAssignment and GuessNode, specifies the described algorithm in detail.

```
GuessCountermodel(⟨({t_1, ..., t_m}, {t})⟩)
  S ← (s_1, ..., s_n)  ▷ s_i ∈ subt({t_1, ..., t_m, t}) ∪ {⊤} for i = 1, ..., n,
  B ← ((V_1 ← ∅, V_{1,∃} ← ∅, V_{1,∀} ← ∅), ..., (V_n ← ∅, V_{n,∃} ← ∅, V_{n,∀} ← ∅))
  for i ← 1 to n
    if (s_i ∈ {t_1, ..., t_m})
      V_1 ← V_1 ∪ {x_i = 1}
    else if (s_i = t)
      V_1 ← V_1 ∪ {x_i < 1}
    endif
  endfor
  b ← 0  ▷ traversal step counter
  j ← 1, d ← 1  ▷ visiting node at distance j − 1 from the root, backtracking if d = 0
  repeat
    b ← b + 1
    if (d = 0 and V_{j,∃} = ∅)
      j ← j − 1, d ← 0
    else if (d = 0 and V_{j,∃} ≠ ∅)
      j ← j + 1, d ← 1
      if (j > n)
        output 0
      else
        output GuessNode(j, B)
      endif
    else if (d = 1)
      if not(GuessAssignment(j, B))
        output 0
      else if(V_{j,∃} = ∅)
        j ← j − 1, d ← 0
      else if(V_{j,∃} ≠ ∅)
        j ← j + 1, d ← 1
        if (j > n)
          output 0
        else
          output GuessNode(j, B)
        endif
```

```
    endif
  endif
until (j = 0 or b = 2^{q(n)+1} − 1)
if(j = 0)  ▷ traversal terminated
  output 1
else  ▷ step counter out of bound
  output 0
endif
```

GuessAssignment(j, B)

```
guess M_j ≤ 2^{b(n)}, (g_1, ..., g_n) ∈ [M_j + 1]^n
for i ← 1 to n
  if(s_i = ⊥)
    V_j ← V_j ∪ {x_i = 0}
  else if(s_i = ⊤)
    V_j ← V_j ∪ {x_i = 1}
  else if(s_i = s_{i_1} ∧ s_{i_2})
    if(g_{i_1} ≤ g_{i_2})
      V_j ← V_j ∪ {x_i = x_{i_1}}
    else if(g_{i_2} ≤ g_{i_1})
      V_j ← V_j ∪ {x_i = x_{i_2}}
    endif
  else if(s_i = s_{i_1} ∨ s_{i_2})
    if(g_{i_1} ≤ g_{i_2})
      V_j ← V_j ∪ {x_i = x_{i_2}}
    else if(g_{i_2} ≤ g_{i_1})
      V_j ← V_j ∪ {x_i = x_{i_1}}
    endif
  else if(s_i = s_{i_1} ⊙ s_{i_2})
    if(g_{i_1} + g_{i_2} ≤ 1)
      V_j ← V_j ∪ {x_i = 0}
    else if(g_{i_1} = g_{i_2} = 1)
      V_j ← V_j ∪ {x_i = 1}
    else if(1 < g_{i_1} + g_{i_2})
      V_j ← V_j ∪ {x_i = x_{i_1} + x_{i_2} − 1}
    endif
  else if(s_i = s_{i_1} → s_{i_2})
    if(g_i = 0)
      if(g_{i_1} = 1 and g_{i_2} = 0)
        guess r ∈ {0, 1}
        if(r = 0)
          V_j ← V_j ∪ {x_{i_2} = 0, x_{i_1} = 1}
          V_{j,∀} ← V_{j,∀} ∪ {x_{i_1} ≤ x_{i_2}}
        else
          V_{j,∃} ← V_{j,∃} ∪ {x_{i_2} < x_{i_1}}
        endif
      else
        V_{j,∃} ← V_{j,∃} ∪ {x_{i_2} < x_{i_1}}
      endif
    else if(0 < g_i)
      V_{j,∀} ← V_{j,∀} ∪ {x_{i_1} ≤ x_{i_2}}
      if(g_{i_1} ≤ g_{i_2})
        V_j ← V_j ∪ {x_{i_1} ≤ x_{i_2}}
      else if(0 = g_{i_2} < g_{i_1} < 1)
        V_j ← V_j ∪ {x_i = 1 − x_{i_1}}
      else if(0 < g_{i_2} < g_{i_1} = 1)
        V_j ← V_j ∪ {x_i = x_{i_2}}
      else if(0 < g_{i_2} < g_{i_1} < 1)
        V_j ← V_j ∪ {x_i = x_{i_2} + 1 − x_{i_1}}
      endif
    endif
  endif
  if(0 < g_i)
    V_j ← V_j ∪ {0 < x_i}
    V_{j,∀} ← V_{j,∀} ∪ {x_i = 1}
  endif
endfor
```

```
58 if(x_1 ↦ g_1, ..., x_n ↦ g_n satisfies V_j)
59    output true
60 else
61    output false
62 endif
```

```
GuessNode(j, B)
1  guess F ⊆ V_{j−1,∃}, F ≠ ∅
2  forall (i_1, i_2) ∈ {1, ..., n}^2
3     if(x_{i_2} < x_{i_1} ∈ F)
4        V_j ← V_j ∪ {0 < x_{i_1}, x_{i_2} < x_{i_1}, x_{i_2} < 1}
5        for k ← 1 to j − 1
6           V_{k,∃} ← V_{k,∃} \ F
7        endfor
8     endif
9     if(x_{i_1} ≤ x_{i_2} ∈ V_{j−1,∀})
10       V_j ← V_j ∪ {x_{i_1} ≤ x_{i_2}}
11       V_{j,∀} ← V_{j,∀} ∪ {x_{i_1} ≤ x_{i_2}}
12    endif
13    if(x_{i_1} = 1 ∈ V_{j−1,∀})
14       V_j ← V_j ∪ {x_{i_1} = 1}
15       V_{j,∀} ← V_{j,∀} ∪ {x_{i_1} = 1}
16    endif
17 endforall
```

On Line 1 of GuessCountermodel, the input $E = (\{t_1, \ldots, t_m\}, \{t\})$, such that $|\langle E \rangle| = n$ and $\mathsf{var}(E) = \{y_1, \ldots, y_l\}$, is parsed into a tuple S of the form $(s_1, \ldots, s_n)$, containing all the subterms of E. W.l.o.g. we assume that: $s_1 = y_1, \ldots, s_l = y_l$; $\mathsf{op}(s_{l+1}) \leq \mathsf{op}(s_{l+2}) \leq \cdots \leq \mathsf{op}(s_{|\mathsf{subt}(E)|})$, breaking ties lexicographically; $s_{|\mathsf{subt}(E)|+1} = \cdots = s_n = \top$. Recall that $t_1, \ldots, t_m, t \in \mathsf{subt}(E)$, thus $t_1, \ldots, t_m, t$ are items of S.

The procedure GuessCountermodel maintains in memory a bounded LIFO stack B of n items to store the nodes in the current path (Line 2, Lines 19–20 and Lines 31–32). The jth item of B, $j \leq n$, is a triple $(V_j, V_{j,\forall}, V_{j,\exists})$ of sets of linear equality and inequality constraints over the variables $x_1, \ldots, x_n$, *representing* the node v at distance $j-1$ from the root along the path currently in memory, in the following sense. The set V_j represents the constraints that an assignment $\mathbf{g} = (g_1, \ldots, g_n) \in [M_j + 1]^n$ ($M_j \leq 2^{b(n)}$) of variables $x_1, \ldots, x_n$ onto $[M_j + 1]$, corresponding to the node v, must satisfy, in order to verify the following conditions:

(i) The assignment $\mathbf{g}$ is *consistent* with Definition 2, that is, the assignment $y_1 \mapsto g_1, \ldots, y_l \mapsto g_l$ of $\mathsf{var}(E)$ over $[M_j + 1]$ extends to a valuation of the subterms $s_{l+1}, \ldots, s_{|\mathsf{subt}(E)|} \in \mathsf{subt}(E) \setminus \{y_1, \ldots, y_l\}$ such that $(s_{l+1})_{(g_1,\ldots,g_l)} = g_{l+1}, \ldots, (s_{|(E)|})_{(g_1,\ldots,g_l)} = g_{|\mathsf{subt}(E)|}$. This condition is checked by GuessAssignment, as follows: on Lines 2–57, for every subterm s in S, V_j is enriched with constraints ensuring that $x_1 \mapsto g_1, \ldots, x_n \mapsto g_n$ is a solution to V_j if and only if $\mathbf{g}$ is consistent with Definition 2; finally, on Line 58, the consistency of $\mathbf{g}$ is tested, outputting **true** if and only if the outcome is positive. In addition, GuessAssignment memorizes in $V_{j,\forall}$ the universal constraints pending on v with respect to $(g_1, \ldots, g_l)$ (Lines 33 and 41 and Line 55 respectively), and in $V_{j,\exists}$ the existential constraints pending on v with respect to $(g_1, \ldots, g_l)$ (Lines 35 and 38).

(ii) If v is the root node of $\mathbf{T}$ (that is, $j = 1$), then the assignment $\mathbf{g}$ is such that $t_{(g_1,\ldots,g_l)} < 1 = (t_1)_{(g_1,\ldots,g_l)} = \cdots = (t_m)_{(g_1,\ldots,g_l)}$ holds in v. This condition is preliminary imposed over V_1 by Lines 3–9 of GuessCountermodel.

(iii) If v is an internal node of $\mathbf{T}$ (that is, $j > 1$), then all the inherited universal constraints hold in v with respect to $(g_1, \ldots, g_l)$, and $(g_1, \ldots, g_l)$ satisfies a nonempty set F of inherited existential constraint. The former condition is imposed over V_j by Lines 10 and 14 of GuessNode(j, B). Note also that Lines 11 and 15 memorize universal constraints on v. The latter condition is imposed over V_j by Lines 1 and 3–4 of GuessNode(j, B). Note also that Line 6 subtracts F from the sets of pending existential constraints on nodes at distance $\leq j-1$ along the path currently in memory.

Overall, the procedure works as follows. At step $b = 1$ (Line 13), the algorithm creates a node r from which to start the path, and guesses an assignment $(g_1, \ldots, g_l) \in [M_1 + 1]^l$ ($M_1 \leq 2^{b(n)}$) to the variables in $\mathsf{var}(E)$ such that $(t)_{(g_1,\ldots,g_l)} < 1 = (t_1)_{(g_1,\ldots,g_l)} = \cdots = (t_m)_{(g_1,\ldots,g_l)}$. In addition, the algorithm memorizes the (universal and existential) constraints pending on r with respect to $(g_1, \ldots, g_l)$. Now, let v be the current node at step $b \geq 1$ (Line 13). There are two cases. Either v has pending existential requirements (GuessCountermodel, Line 16, 24), or not (GuessCountermodel, Line 14, 22). Case 1: GuessNode creates a node u, successor of v, and guesses an assignment corresponding to u such that u satisfies at least one existential constraint pending on v. Every existential constraint satisfied by u is removed from the existential constraints pending on the ancestors of u (GuessNode, Line 1 and Lines 3–7). In addition, u inherits all the universal constraints propagated by its ancestors (Lines 9–12 and 13–16). The procedure iterates over u. Case 2: The visit backtracks to the ancestor w of v. If $w = r$ the algorithm terminates, otherwise the procedure iterates over w. After at most $2^{q(n)+1} - 1$ iterations of the main loop, the procedure terminates (Lines 10, 13 and 29).

Notice that our decision algorithm can be easily translated into a search algorithm, outputting a countermodel to E if E is not valid, without affecting its space complexity. Indeed, in general a countermodel has size exponential in n, but the memory storage for outputting is not metered.

In the next two sections, inspecting the pseudocode, we study the correctness and complexity of our algorithm.

3.1.1. Correctness lemma

In this section, we prove that our algorithm is correct, that is, the algorithm terminates with output 1 if and only if the input quasiequation is not valid.

Lemma 3. GuessCountermodel$(\langle E\rangle) = 1$ *if and only if E is not valid.*

Proof. The algorithm terminates, since the main procedure terminates after at most $2^{q(n)+1} - 1$ iterations (GuessCountermodel, Line 29) and each of the two subprocedures terminates. Let $n = |\langle E\rangle|$.

($\Leftarrow$) Suppose that E is not valid. We prove that there exists a sequence of guesses leading GuessCountermodel to output 1. Let $\mathbf{B}$ be a finite $2^{b(n)}$-bounded poset, determined as in Lemma 2, where E fails, and let $\mathbf{k} = (k_1, \ldots, k_l) \in B^l$ such that E fails in $\mathbf{B}$ with respect to $\mathbf{k}$ and r. By direct inspection of the pseudocode, it is immediate to realize that if the sequence of nodes guessed by the algorithm one-to-one corresponds to the sequence of nodes visited during a preorder traversal of $\mathbf{T}$, and the assignment $\mathbf{g} = (g_1, \ldots, g_n)$ guessed over any node v, corresponding to the node $u \in T$, satisfies: $g_1 = (y_1)_{\mathbf{k},u}$, $\ldots, g_l = (y_l)_{\mathbf{k},u}, g_{l+1} = (s_{l+1})_{\mathbf{k},u}, \ldots, g_{|\mathrm{subt}(E)|} = (s_{|\mathrm{subt}(E)|})_{\mathbf{k},u}, g_{|\mathrm{subt}(E)|+1} = 1, \ldots, g_n = 1$, then after at most $2^{q(n)+1} - 1$ steps the main loop terminates with $j = 0$ and the algorithm outputs 1 (GuessCountermodel, Line 31). Indeed, $2^{q(n)+1} - 1$ steps suffice to complete the preorder traversal of $\mathbf{T}$, which is a tree of cardinality at most $2^{q(n)}$, and a preorder traversal of a rooted tree starts and terminates on the root of the tree.

($\Rightarrow$) Suppose that GuessCountermodel outputs 1 on input E. By direct inspection of the pseudocode, it is immediate to realize that an execution GuessCountermodel outputting 1 is equivalent to preorder traverse a finite tree $\mathbf{T} = (T, E_T)$ rooted at r, and to compute a tuple $\mathbf{k} = (k_1, \ldots, k_l)$ of functions ($k_i(u) \in [M_u + 1]$ for every $u \in T$, where $i = 1, \ldots, l$ and $M_u \leq 2^{b(n)}$) such that, letting $\mathbf{B}$ be the $2^{b(n)}$-bounded poset sum with skeleton $\mathbf{T}$, E fails in $\mathbf{B}$ with respect to r and $\mathbf{k}$. Then, E is not valid. □

3.1.2. Space bound

In this section, we prove that our algorithm allocates an amount of memory bounded above by a polynomial of the size, n, of the input. To this aim, we exploited Lemma 2 to reduce the search space to $2^{b(n)}$-bounded poset sums, having as skeletons rooted trees of height at most n and cardinality at most $2^{q(n)}$.

Lemma 4. $\overline{\text{GBL-CB-QEQ}} \in$ **NPSPACE**.

Proof. For any possible sequence of guesses, inspecting the pseudocode, we observe that memory space is allocated to store the following data structures: the list S of the n subterms of the input terms $t_1, \ldots, t_m, t$ (GuessCountermodel, Line 1); the list B, containing n triples $(V_j, V_{j,\forall}, V_{j,\exists})$, where V_j is a set of at most $2n^3 + 6n^2 + 2n$ linear constraints over n variables, and $V_{j,\forall}, V_{j,\exists}$ are sets of at most n^2 linear constraints over n variables (GuessCountermodel, Line 2); the step counter b, ranging over nonnegative integers $\leq 2^{q(n)+1} - 1$ (GuessCountermodel, Line 10); a constant number of counters/variables, ranging over nonnegative integers $\leq n$; the integer $M_j \leq 2^{b(n)}$ and the tuple $(g_1, \ldots, g_n) \in [M_j + 1]$ (GuessAssignment, Line 1); the random bit r (on Line 31 of GuessAssignment); the set F, containing at most n^2 linear constraints over n variables (GuessNode, Line 1). For any reasonably compact encoding of the objects involved (integers, pairs, tuples, sets, etc.), each of these data structures requires an amount of space polynomial in n to be stored, therefore, an amount of space polynomial in n suffices to store simultaneously a constant number of the structures described. Moreover, all the subprocedures invoked (for analyzing a term into subterms, checking if a term is member of a finite set of terms or is equal to another term, adding elements to sets, removing elements from finite sets, checking if a linear constraint is satisfied under a variables assignment) receive in input the structures described above and work in time polynomial in the input size, hence they can be executed in space polynomial in n. Overall, an amount of space polynomial in n suffices to execute the algorithm.

Thus, the nondeterministic algorithm GuessCountermodel works in polynomial space independent of the guesses made, satisfying clause (ii) of Definition 5. Since, by Lemma 3, GuessCountermodel satisfies also clause (i) of Definition 5, we conclude that $\overline{\text{GBL-CB-QEQ}} \in$ **NPSPACE**. □

Corollary 1. GBL-CB-QEQ $\in$ **PSPACE**.

Proof. By Lemma 4, GBL-CB-QEQ is in **coNPSPACE**. But **coNPSPACE** = **NPSPACE** and **NPSPACE** = **PSPACE** [19]. □

3.2. Lower bound

We conclude by showing that GBL-CB-QEQ is hard for **PSPACE**. This hardness result provides evidence that, in the general case, if a quasiequation E is not in GBL-CB-QEQ, any object witnessing failure must have size at least exponential in the size of n.

Lemma 5. GBL-CB-QEQ *is* **PSPACE**-*hard.*

Proof. The problem INT-TAUT, of deciding if a propositional formula ϕ over $\mathcal{L}_1 \setminus \{\odot\}$ and $\{y_1, \ldots, y_l\}$ is intuitionistically provable (say, in the intuitionistic natural deduction calculus), is **PSPACE**-complete [21]. Hence, to prove the lemma, we describe a polynomial-time reduction that receives in input an instance $\langle\phi\rangle$ of INT-TAUT and returns in output an instance $\langle E\rangle$ of GBL-CB-QEQ such that $\langle\phi\rangle \in$ INT-TAUT if and only if $\langle E\rangle \in$ GBL-CB-QEQ.

Every propositional formula ϕ over $\mathcal{L}_1 \setminus \{\odot\}$ containing variables among $y_1, \ldots, y_l$ *corresponds* to a term t over $\mathcal{L}_1 \setminus \{\odot\}$ containing variables among $y_1, \ldots, y_l$, under the obvious mapping. For any algebra $\mathbf{A}$ over $\mathcal{L}_1$, having domain A, we write $t^{\mathbf{A}}$ for the l-variate operation over A corresponding to the term t. Let $\mathbf{H}_l$ be the free l-generated Heyting algebra. $\mathbf{H}_l$ is isomorphic to the Lindenbaum-Tarski algebra of intuitionistic propositional formulas over l variables [20]: thus, if t corresponds to ϕ, $\langle\phi\rangle \in$ INT-TAUT if and only if $t^{\mathbf{H}_l} = \top^{\mathbf{H}_l}$ holds in $\mathbf{H}_l$. Now, let $\phi(y_1, \ldots, y_l)$ be any propositional formula over $\mathbf{L}_1 \setminus \{\odot\}$, and let t be its corresponding algebraic term. Writing for short x^2 instead of $x \odot x$, and $x_1 \leftrightarrow x_2$ instead of $(x_1 \to x_2) \wedge (x_2 \to x_1)$, we put:

$$E \risingdotseq (\{(y_1 \leftrightarrow (y_1)^2) \wedge \cdots \wedge (y_l \leftrightarrow (y_l)^2)\}, \{t\}),$$

and we claim that $\langle\phi\rangle \in$ INT-TAUT if and only if $\langle E\rangle \in$ GBL-CB-QEQ. Clearly, E is polynomial-time computable in the size of the input ϕ.

Suppose that $\langle\phi\rangle \in$ INT-TAUT. Hence, $t^{\mathbf{H}_l} = \top^{\mathbf{H}_l}$ holds in $\mathbf{H}_l$, so that $t^{\mathbf{A}} = \top^{\mathbf{A}}$ holds in every Heyting algebra $\mathbf{A}$, by universal algebra [16]. Now, we exploit the following key fact [13]: if $\mathbf{B}$ is a commutative and bounded *GBL*-algebra, then the subalgebra $\mathbf{A}$ of $\mathbf{B}$, formed by the idempotents of $\mathbf{B}$, is a Heyting algebra. Therefore, $t^{\mathbf{A}} = \top^{\mathbf{A}}$ holds in $\mathbf{A}$. Therefore, since the identity

$$((y_1 \leftrightarrow (y_1)^2) \wedge \cdots \wedge (y_l \leftrightarrow (y_l)^2))^{\mathbf{B}} = \top^{\mathbf{B}} \tag{10}$$

holds in $\mathbf{B}$ if and only if all the variables in t are assigned over idempotent elements of $\mathbf{B}$, we have that, assuming (10), the identity $t^{\mathbf{B}} = t^{\mathbf{A}}$ holds. Thus, since $\top^{\mathbf{A}} = \top^{\mathbf{B}}$, $t^{\mathbf{B}} = \top^{\mathbf{B}}$ holds in $\mathbf{B}$. Thus, the quasiequation E is valid, $\langle E\rangle \in$ GBL-CB-QEQ. Conversely, suppose that $\langle\phi\rangle \notin$ INT-TAUT. Hence, $t^{\mathbf{H}_l} < \top^{\mathbf{H}_l}$ holds in $\mathbf{H}_l$, that is, there exists an assignment $\mathbf{a}$ of the variables in $\mathbf{H}_l$ such that $t^{\mathbf{H}_l} < \top^{\mathbf{H}_l}$ under $\mathbf{a}$. Now, by definition, $\mathbf{H}_l$ is a commutative bounded *GBL*-algebra $\mathbf{B}$ satisfying the identity $x_1 \odot x_2 = x_1 \wedge x_2$. Thus, on the one hand, the identity

$$((y_1 \leftrightarrow (y_1)^2) \wedge \cdots \wedge (y_l \leftrightarrow (y_l)^2))^{\mathbf{B}} = \top^{\mathbf{B}}$$

holds in $\mathbf{B}$ under any assignment, in particular under $\mathbf{a}$. But, on the other hand, $t^{\mathbf{B}} < \top^{\mathbf{B}}$ under $\mathbf{a}$, so we conclude that the quasiequation E fails in $\mathbf{B}$ and $\langle E\rangle \notin$ GBL-CB-QEQ. □

4. Conclusion

A problem raised by this research is to give a lower bound on the complexity of the *equational* theory of commutative bounded *GBL*-algebras, that is, the problem of deciding quasiequations of the form $(\emptyset, \{t\})$. In logical terms, this is the problem of deciding validity in the logic GBL_{ewf}. We state the full result as a conjecture.

Conjecture 1. *The equational theory of commutative bounded GBL-algebras is* **PSPACE***-hard, hence* **PSPACE***-complete.*

Below, we consider a special subvariety of *GBL*-algebras for which we are able to prove **PSPACE**-completeness for both equations and quasiequations. This subvariety is that of *k-potent* commutative bounded *GBL*-algebras, that is commutative bounded *GBL*-algebras satisfying $x^{k+1} = x^k$, corresponding to the logic GBL_{ewf} plus the k-contraction axiom (A14):

$$\underbrace{\phi \odot \cdots \odot \phi}_{k \text{ times}} \to \underbrace{\phi \odot \cdots \odot \phi}_{k+1 \text{ times}}.$$

Theorem 3. *Both the quasiequational theory and the equational theory of k-potent commutative bounded GBL-algebras are* **PSPACE***-complete. Thus both validity and consequence in GBL_{ewf} plus* (A14) *are* **PSPACE***-complete.*

Proof. For **PSPACE** containment, we use the fact that every k-potent *GBL*-algebra is the poset sum of a family of *MV*-chains with cardinality $\le k+1$ [14]. The algorithm is exactly the algorithm for deciding quasiequations in commutative bounded *GBL*-algebras (in particular, the M_j guessed on Line 1 of GuessAssignment is bounded above by the constant k). For **PSPACE** hardness, simply note that the idempotents of a k-potent commutative bounded *GBL*-algebra are precisely the elements of the form x^k. Thus let $t[x \leftarrow x^k]$ denote the term obtained replacing each variable x by x^k in the term t. Since the idempotents of a GBL_{ewf}-algebra constitute a Heyting algebra, we have that $t = \top$ holds in all Heyting algebras if and only if $t[x \leftarrow x^k] = \top$ holds in all k-potent commutative bounded *GBL*-algebras. This yields a reduction from provability in intuitionistic logic *IL* to the validity of equations in k-potent commutative bounded *GBL*-algebras, and the claim follows. □

We conclude this section presenting partial complexity results on the subvarieties of commutative and integral *GBL*-algebras, and commutative *GBL*-algebras, corresponding respectively to the logic GBL_{ew} (that is, GBL_{ewf} minus axiom (A13)), and to the logic GBL_e (that is, GBL_{ewf} minus axioms (A4), (A13) and plus the rule: $A, B \vdash_e a \wedge B$). As regards to these subvarieties we can prove **PSPACE**-completeness of the quasiequational theory, but again the reduction technique does not generalize to the equational case.

Theorem 4. *The following statements hold.*

(i) *The quasiequational theory of commutative and integral GBL-algebras is* **PSPACE***-complete. Thus consequence in GBL_{ew} is* **PSPACE***-complete.*
(ii) *The quasiequational theory of commutative GBL-algebras is* **PSPACE***-complete. Thus consequence in GBL_e is* **PSPACE***-complete.*

Proof. (i) For the upper bound part, first observe that from any commutative integral *GBL*-algebra $\mathbf{A}$ we can obtain a commutative, integral and bounded *GBL*-algebra $\mathbf{B}$ such that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$: just add a new element $\bot$ and extend the operations letting, for every $x, y \in A$: $x \odot \bot \rightleftharpoons \bot$; $x \wedge \bot \rightleftharpoons \bot$; $x \vee \bot \rightleftharpoons x$; $\bot \to x \rightleftharpoons \top$; $y \to \bot \rightleftharpoons \bot$. It follows that for every quasiequation $E = (\{t_1, \ldots, t_m\}, \{t\})$ in the language of commutative *GBL*-algebras we have that E is valid in all commutative integral *GBL*-algebras if and only if E is valid in all commutative bounded *GBL*-algebras, thus proving that the quasiequational theory of commutative integral *GBL*-algebras is in **PSPACE**.

For the lower bound part, we reduce the quasiequational theory of commutative bounded *GBL*-algebras to the quasiequational theory of commutative integral *GBL*-algebras. Let $E = (T, \{t\})$ be a quasiequation in the language of commutative bounded *GBL*-algebras, where $T = \{t_1, \ldots, t_m\}$. Let x be a variable not occurring in $\mathsf{var}(E)$, and let $t[\bot \leftarrow x]$ denote the result of substituting $\bot$ by x in t, for every term t. For every set U of terms, let $U[\bot \leftarrow x] \rightleftharpoons \{u[\bot \leftarrow x] : u \in U\}$. Let

$$S \rightleftharpoons \{x \to s : s \in \mathsf{subt}(E)[\bot \leftarrow x]\} \cup \{x \to x^2\},$$

where we note that if for some assignment $\mathbf{a} : \mathsf{var}(E) \cup \{x\} \to A$ in a commutative integral *GBL*-algebra $\mathbf{A}$ we have that $s^{\mathbf{A}}(\mathbf{a}) = \top$ for every $s \in S$, then $\mathbf{a}(x)$ is an idempotent element of $\mathbf{A}$ such that $\mathbf{a}(x) \le s^{\mathbf{A}}(\mathbf{a})$ holds for all $s \in \mathsf{subt}(E)[\bot \leftarrow x]$. We claim that the quasiequation E is valid in all commutative bounded *GBL*-algebras if and only if the quasiequation E' defined as follows:

$$E' \rightleftharpoons (T[\bot \leftarrow x] \cup S, \{t\}[\bot \leftarrow x])$$

is valid in all commutative integral *GBL*-algebras. For the right to left direction, just replace x by $\bot$ in $T[\bot \leftarrow x] \cup S$ and in $\{t\}[\bot \leftarrow x]$. Let $S[x \leftarrow \bot]$ denote the result of replacing x by $\bot$ in S. Then we get that the quasiequation $(T \cup S[x \leftarrow \bot], \{t\})$ is valid in all commutative bounded *GBL*-algebras. But $S[x \leftarrow \bot]$ entirely consists of valid equations, therefore the quasi equation $(T, \{t\})$ is also valid. For the other direction, suppose that E' is not valid in some commutative integral *GBL*-algebra $\mathbf{A}$. Then there is an assignment $\mathbf{a} : \mathsf{var}(E) \cup \{x\} \to A$ such that $u^{\mathbf{A}}(\mathbf{a}) = \top$ for all $u \in T[\bot \leftarrow x] \cup S$ and $t^{\mathbf{A}}(\mathbf{a}) < \top$. Now it is easy to check that $\mathbf{a}(x)$ is an idempotent of $\mathbf{A}$ and the set of all elements greater than or equal to $\mathbf{a}(x)$ is a subalgebra $\mathbf{B}$ of $\mathbf{A}$ which contains all elements of the form $s^{\mathbf{A}}(\mathbf{a})$ for $s \in \mathsf{subt}(E)$. Since $\mathbf{a}(x)$ is the bottom of $\mathbf{B}$, we can safely interpret $\bot$ over $\mathbf{a}(x)$, thus getting an assignment into a commutative bounded *GBL*-algebra which invalidates E.

(ii) We already mentioned that every commutative *GBL*-algebra decomposes as a direct product of a commutative and integral *GBL*-algebra and a lattice ordered Abelian group [10]. It follows that a quasiequation holds in all commutative *GBL*-algebras if and only if it holds in all commutative and integral *GBL*-algebras and in all lattice ordered Abelian groups. Since the quasiequational theory of lattice ordered Abelian groups is in **coNP** (**coNP**-complete in fact, [22]), it is in **PSPACE**. On the other hand, the quasiequational theory of commutative integral *GBL*-algebras is in **PSPACE**, therefore we have shown **PSPACE** containment. As regards to **PSPACE** hardness, we reduce the quasiequational theory of commutative integral *GBL*-algebras to the quasiequational theory of commutative *GBL*-algebras. The reduction is based on the following statement [17].

Fact 1. *Let* $\mathbf{A}$ *be a commutative GBL-algebra, with operations* $\cdot, \vee, \wedge, \to$ *and neutral element* e. *Let* $A^- \rightleftharpoons \{a \in A : a \le e\}$. *Define for* $x, y \in A^-$ *and for* $\circ \in \{\cdot, \vee, \wedge\}$, $x \circ^- y \rightleftharpoons x \circ y$. *Moreover define* $x \to^- y \rightleftharpoons (x \to y) \wedge e$. *Then* $\mathbf{A}^- = (A^-, \cdot^-, \vee^-, \wedge^-, \to^-, e)$ *is an integral GBL-algebra. Moreover* $\mathbf{A}$ *is integral if and only if* $\mathbf{A}^- = \mathbf{A}$.

Now define for every term t of commutative *GBL*-algebras, a term t^- by induction as follows: if t is a variable or a constant, then $t^- \rightleftharpoons t \wedge e$; $^-$ commutes with $\odot$, $\vee$ and $\wedge$; $(s \to u)^- \rightleftharpoons (s^- \to u^-) \wedge e$. As usual, let $t^{\mathbf{A}}$ and $t^{\mathbf{A}^-}$ denote the interpretation of t in $\mathbf{A}$ and in $\mathbf{A}^-$ respectively, and let for every quasiequation $E = (\{t_1, \ldots, t_m\}, t)$, $E^{\mathbf{A}}$ and $E^{\mathbf{A}^-}$ denote $(\{t_1^{\mathbf{A}}, \ldots, t_m^{\mathbf{A}}\}, \{t^{\mathbf{A}}\})$ and $(\{t_1^{\mathbf{A}^-}, \ldots, t_m^{\mathbf{A}^-}\}, \{t^{\mathbf{A}^-}\})$ respectively. Also, let E^- denote the quasiequation $(\{t_1^-, \ldots, t_m^-\}, \{t^-\})$.

Claim 3. *Let* t *be a term with* $\mathsf{var}(t) = k$. *The following statements hold.*

(i) *For all* $a_1, \ldots, a_k \in A$, $(t^-)^{\mathbf{A}}(a_1, \ldots, a_k) \in A^-$.
(ii) *For all* $a_1, \ldots, a_k \in A^-$, $(t^-)^{\mathbf{A}}(a_1, \ldots, a_k) = t^{\mathbf{A}^-}(a_1, \ldots, a_k)$.
(iii) *A quasiequation* E *is valid in* $\mathbf{A}^-$ *if and only if* E^- *is valid in* $\mathbf{A}$.

Proof. (i) and (ii) are shown by a straightforward induction on the term t, and (iii) follows from (ii). □

Claim 4. *A quasiequation* E *holds in all commutative integral GBL-algebras if and only if* E^- *holds in all commutative GBL-algebras.*

Proof. If E fails in some commutative integral *GBL*-algebra $\mathbf{A}$, then by Claim 3(iii), E^- fails in $\mathbf{A}^- = \mathbf{A}$, and therefore it fails in some commutative *GBL*-algebra. Conversely, if E^- fails in some commutative *GBL*-algebra $\mathbf{A}$, then by Claim 3(iii), E fails in $\mathbf{A}^-$, therefore it fails in some commutative integral *GBL*-algebra. □

We conclude the proof of the theorem. Claim 4 shows that the set of quasiequations valid in all commutative integral *GBL*-algebras is reducible in polynomial time to the set of quasiequations which are valid in all commutative *GBL*-algebras, therefore this last set is **PSPACE**-hard. We have already shown that it is in **PSPACE**, therefore it is **PSPACE**-complete. □

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